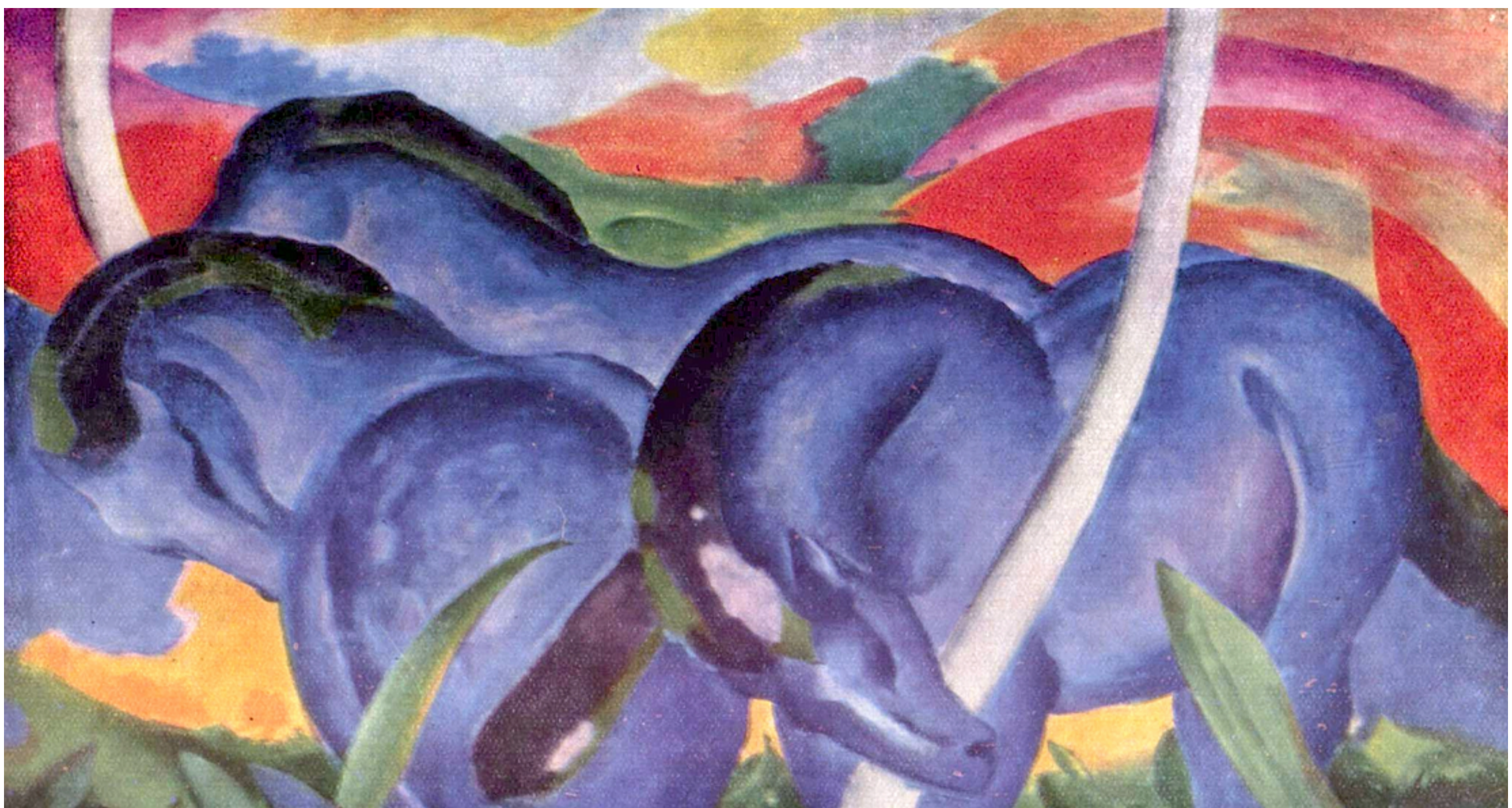
QUANTIZATION OF THREE-BODY SCATTERING AMPLITUDE

Maxim Mai
The George Washington University



Franz Marc, "Die grossen blauen Pferde" 1911

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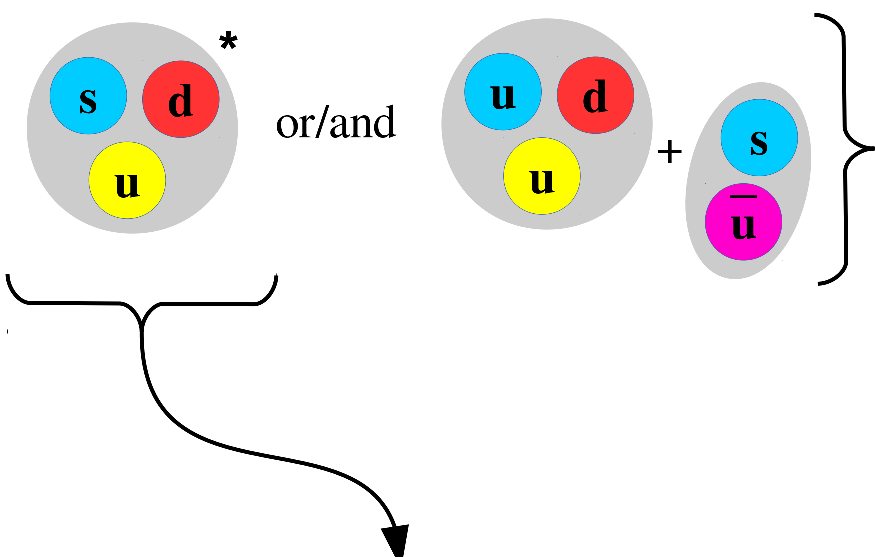
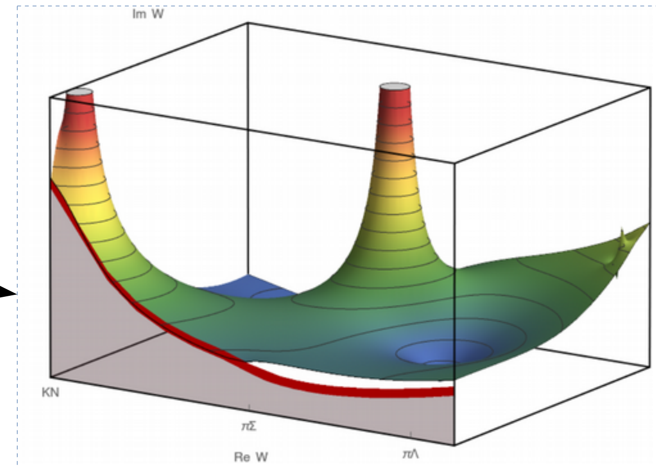
E.g.: Λ(1405)





MM Meissner (2015)
Cieply, MM, Meissner, Smejkal (2016)

first Lattice QCD study: 10% of total W.F.
Hall et al. (2014)

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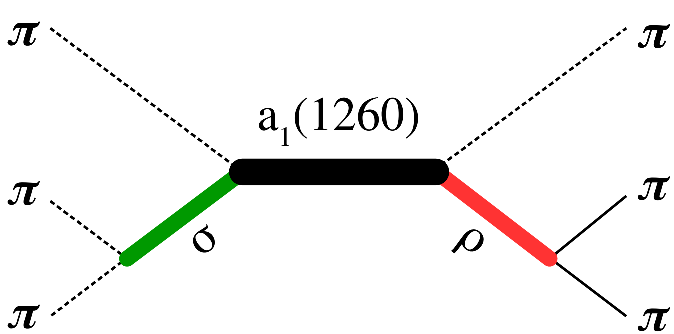

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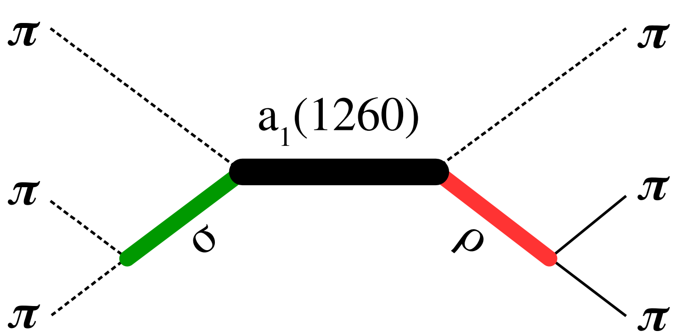

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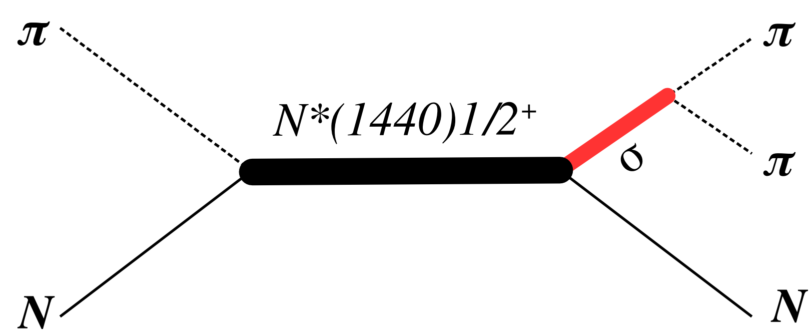

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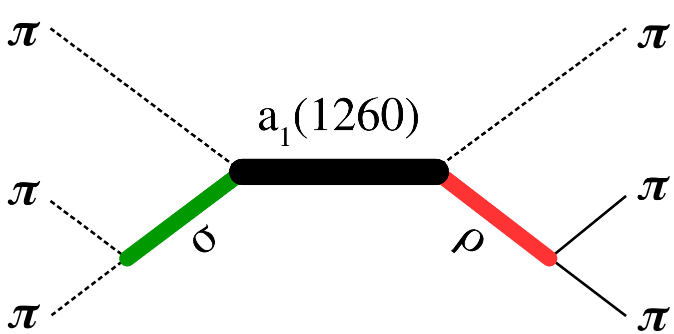

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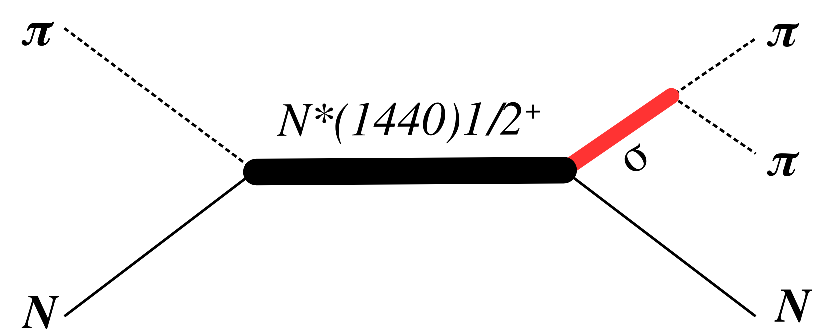
GLUEX citations periment



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- first Lattice QCD results:

w. incomplete treatment of *ππN*

→ NO Roper-signal **Lang et al. (2017)**

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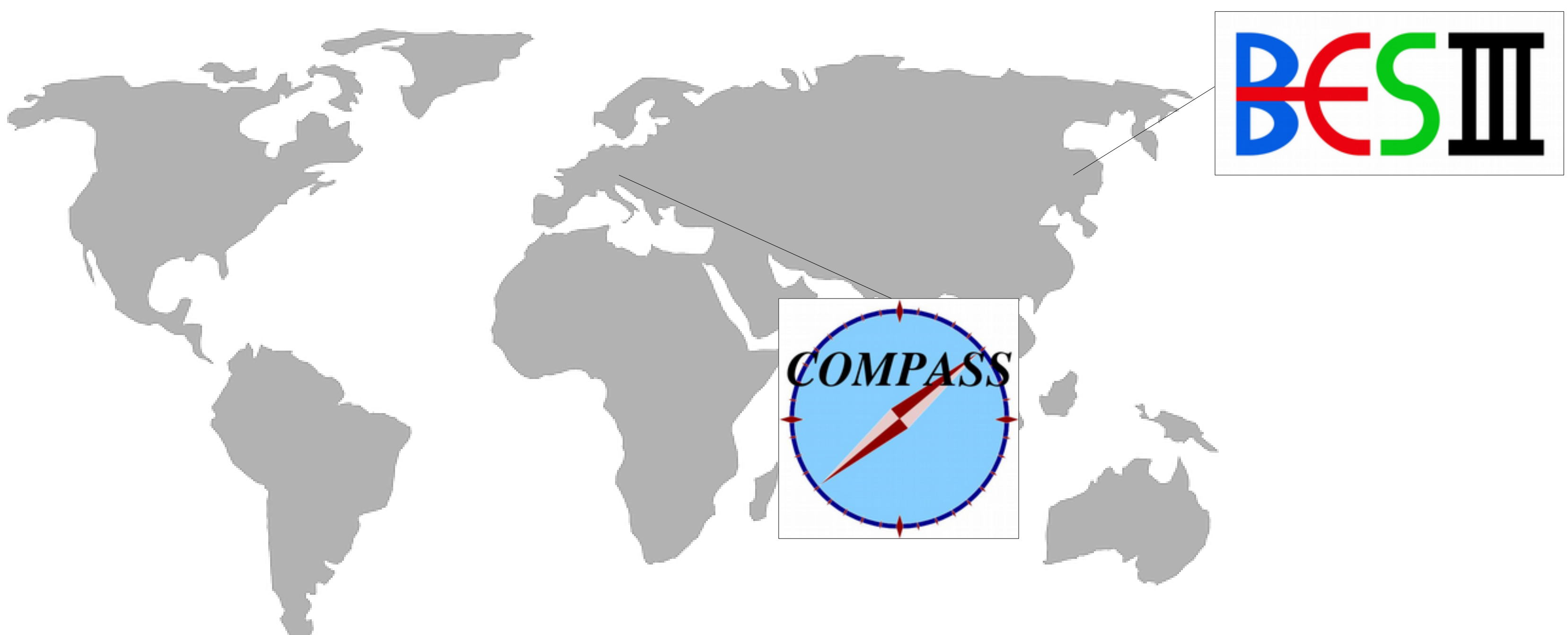

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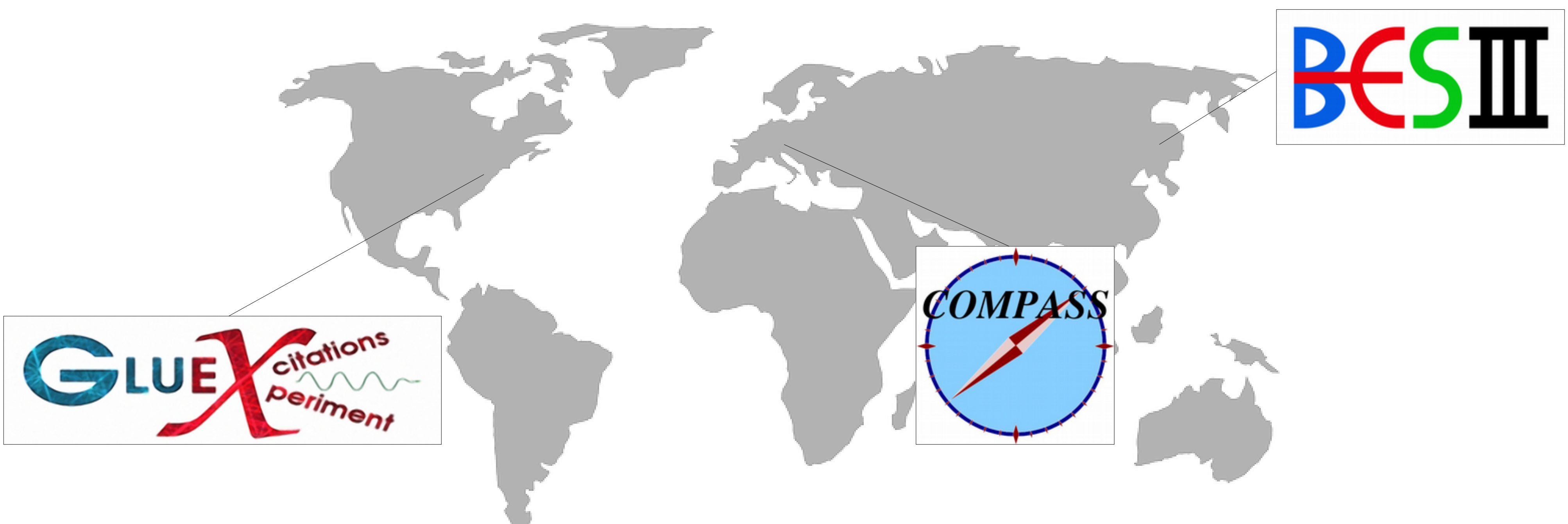

Universal understanding of Lattice QCD or experimental searches (*BESIII, COMPASS, GlueX*)

→ theory of 3-body scattering problem

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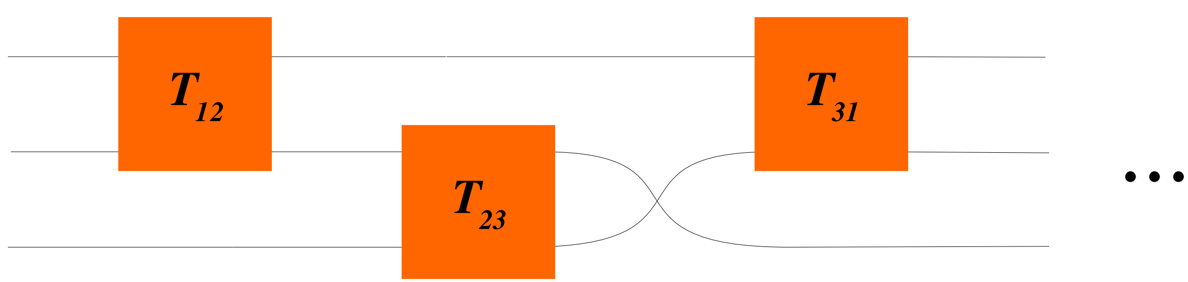

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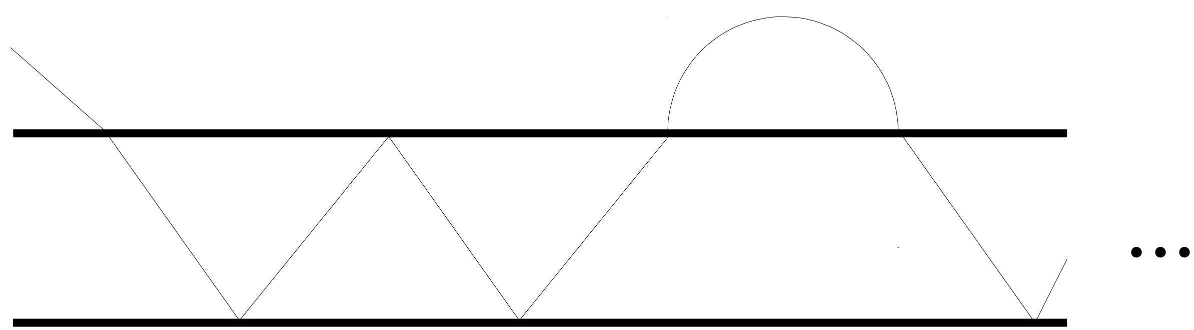

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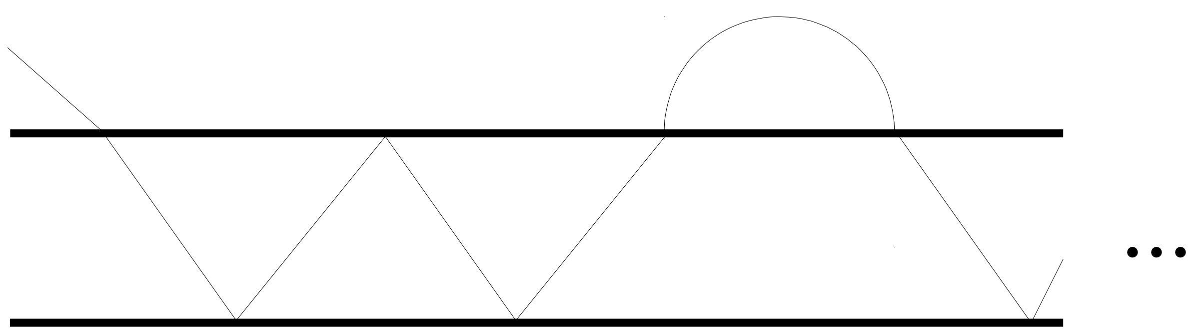

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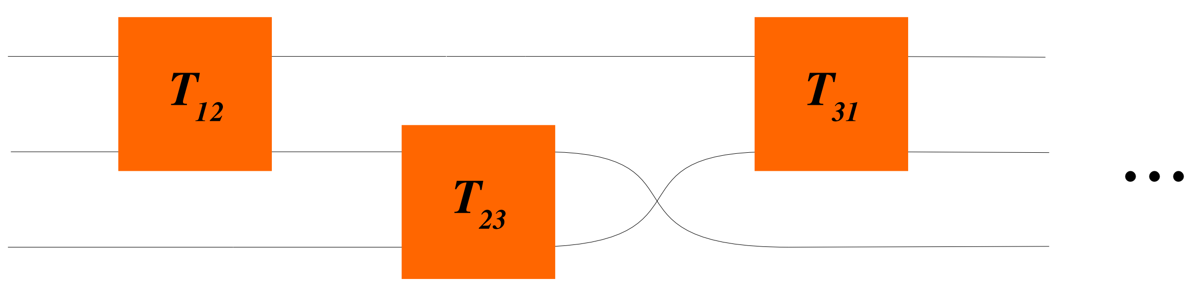

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  - → re-parametrization of two-body amplitude — **Bedaque(1999)**

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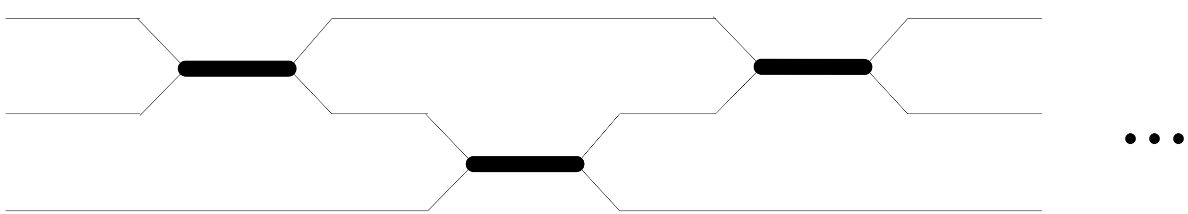

FADDEEV EQUATIONS WITH ISOBARS

MM, Hu, Döring, Pilloni, Szczepaniak

Eur.Phys.J. A53 (2017) no.9, 177





FE in isobar parametrization

Original study – Amado Model **Amado,Aaron,Young(1968)**

- 3-dimensional integral equation from unitarity constraint & BSE ansatz
- valid below break-up energies ($E < 3m$) & analyticity constraints unclear

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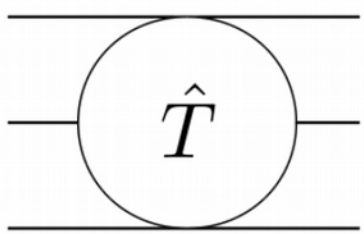
One has to begin with asymptotic states



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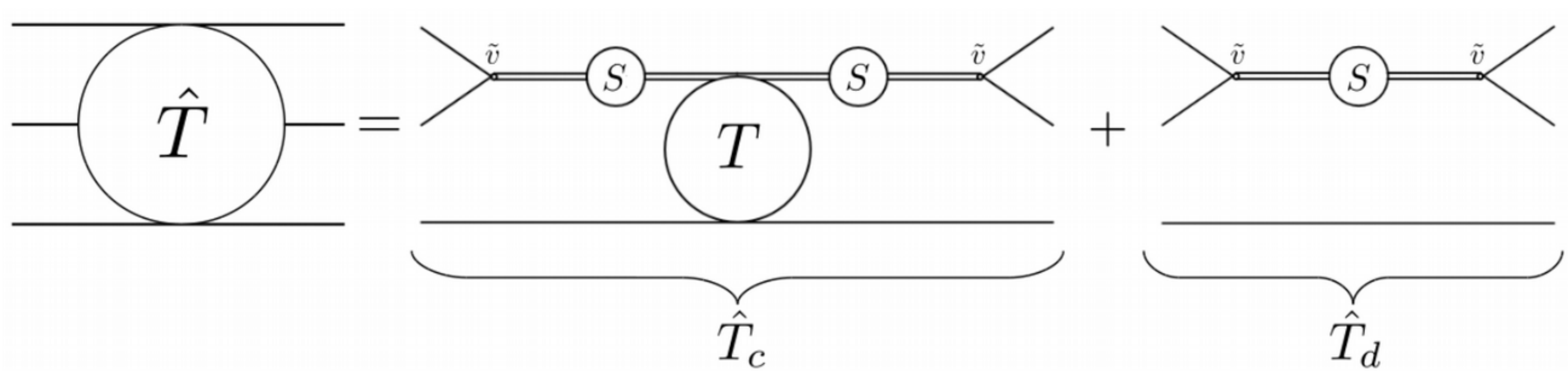

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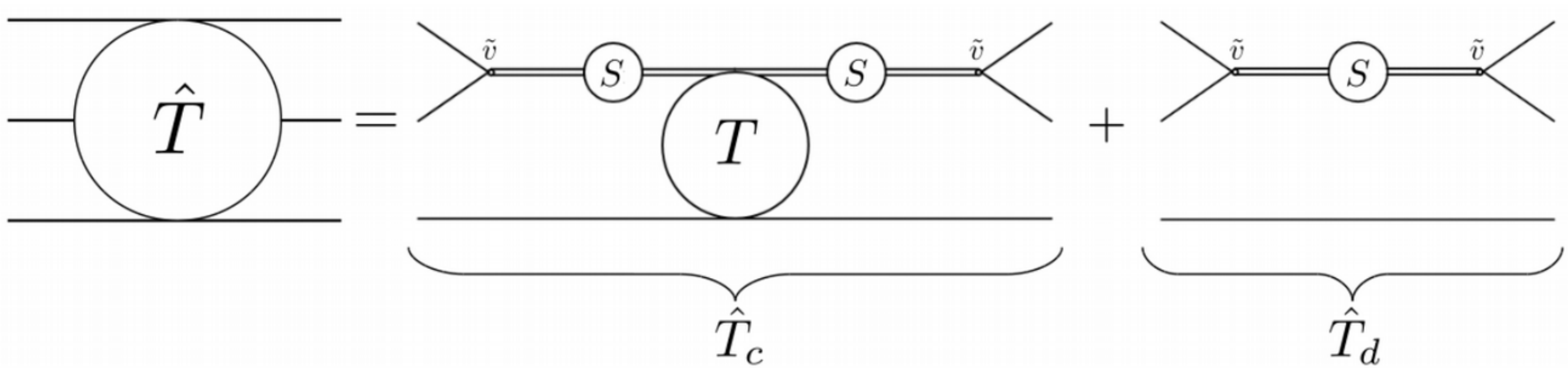

- **two-body interaction is parametrized by an *"isobar"***

= has definite QN and correct r.h.-singularities w.r.t invariant mass

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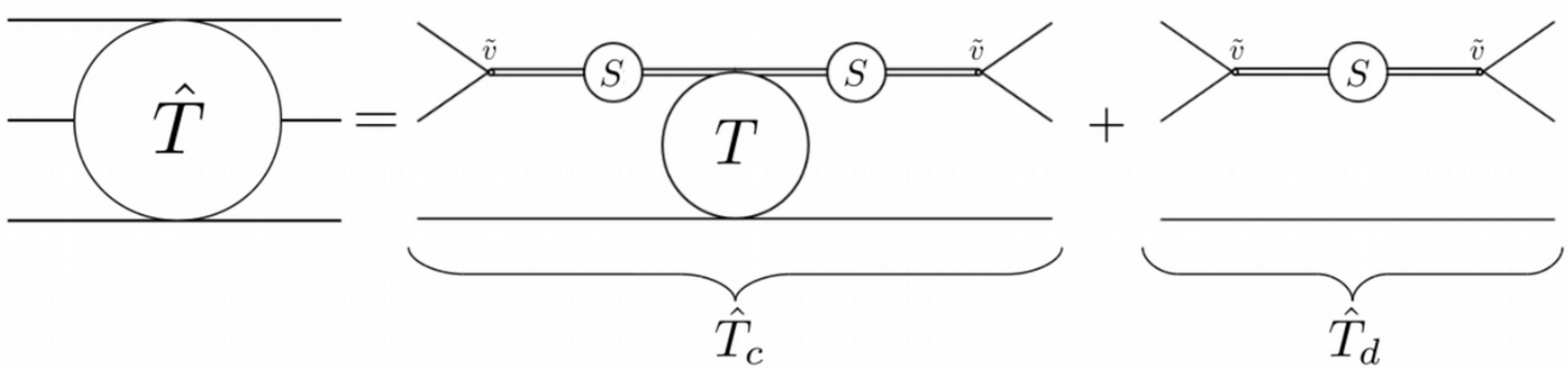

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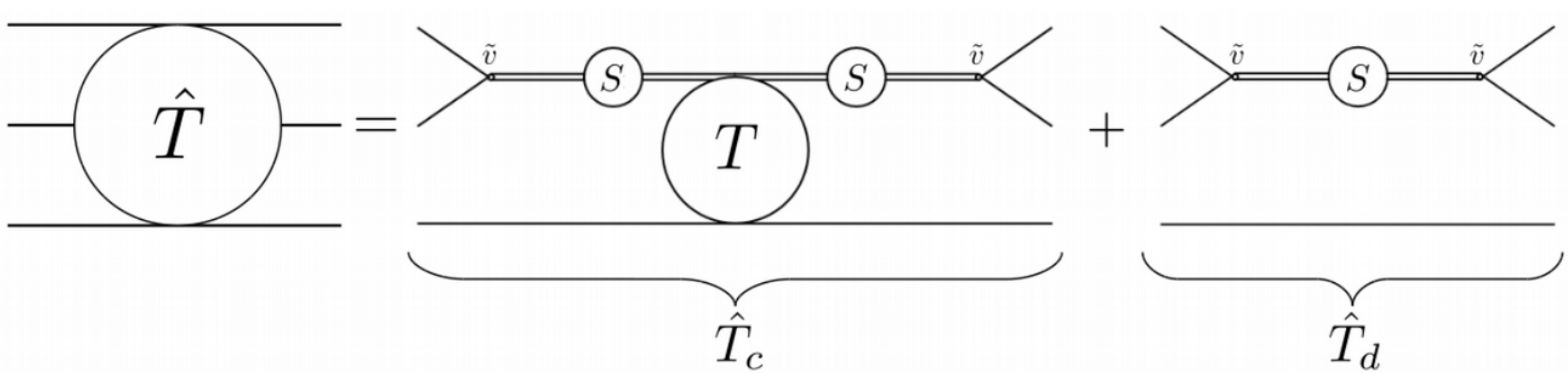

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- ***v*** **a general function** **<u>without cuts in the phys. region</u>**

Unitarity & Matching

3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle \quad = \quad i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$

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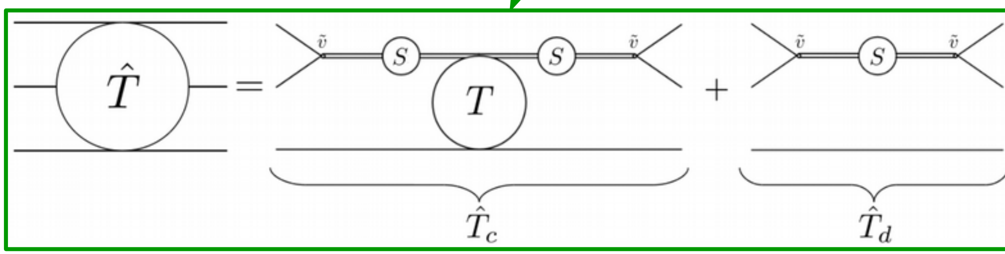

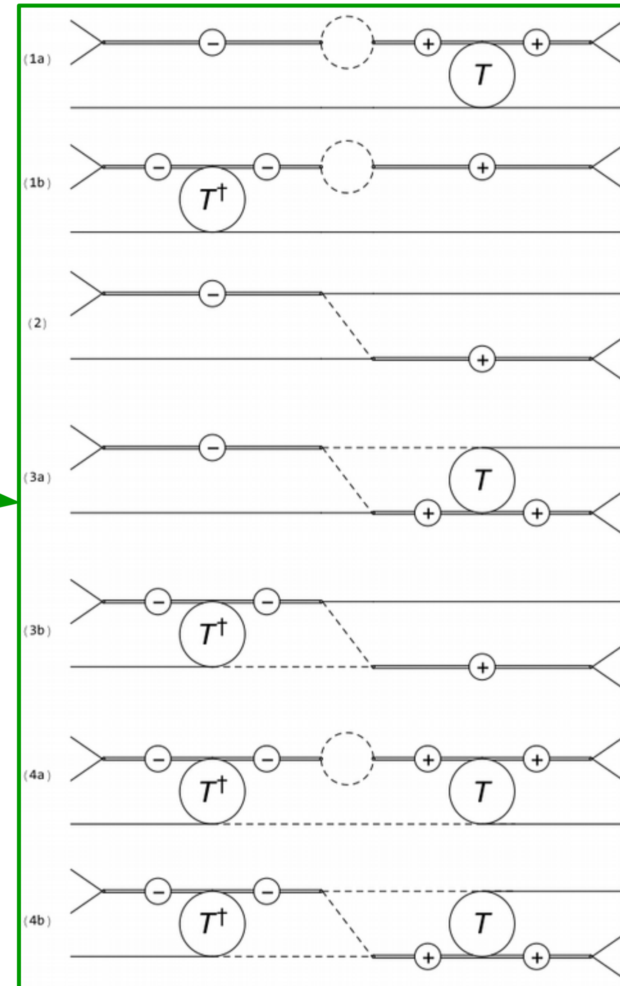

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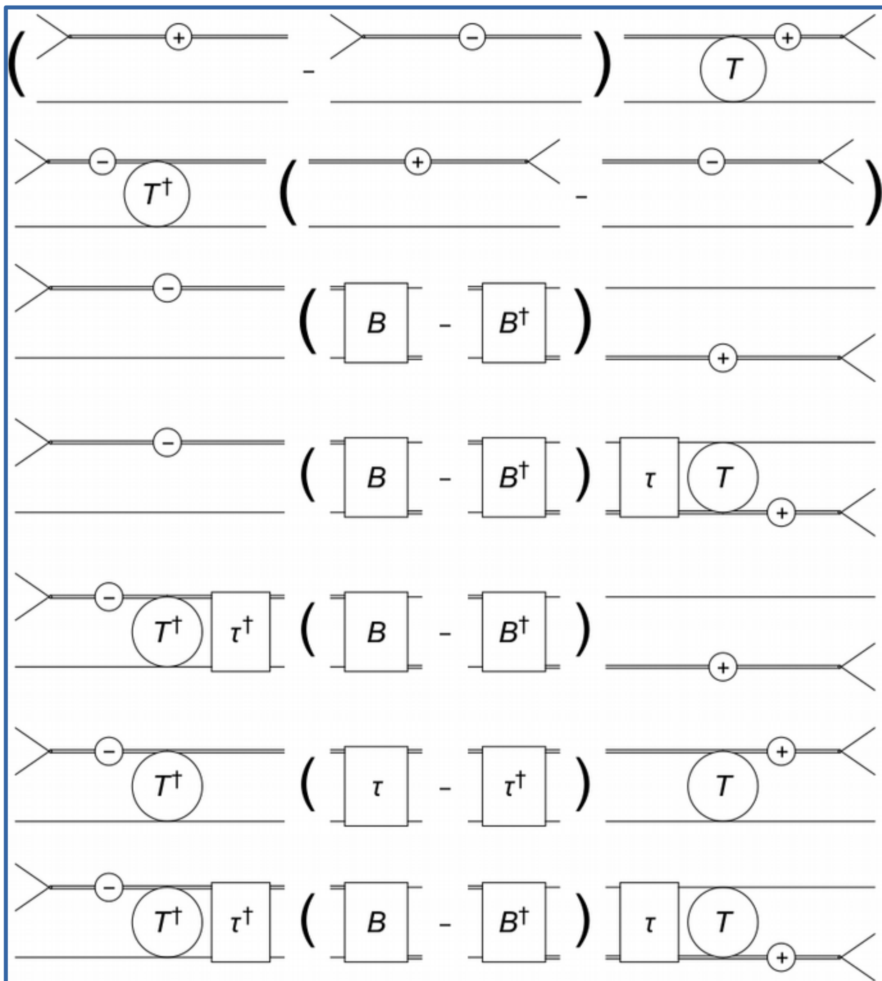

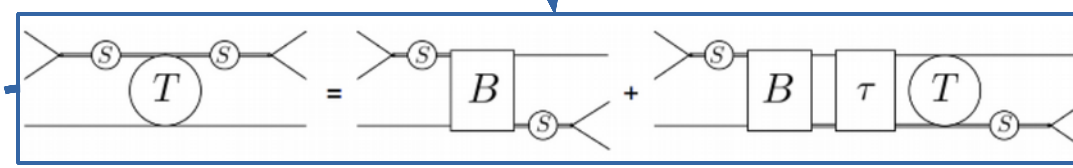

General ansatz for the Isobar-spectator interaction
→ B & τ are unknown!!!

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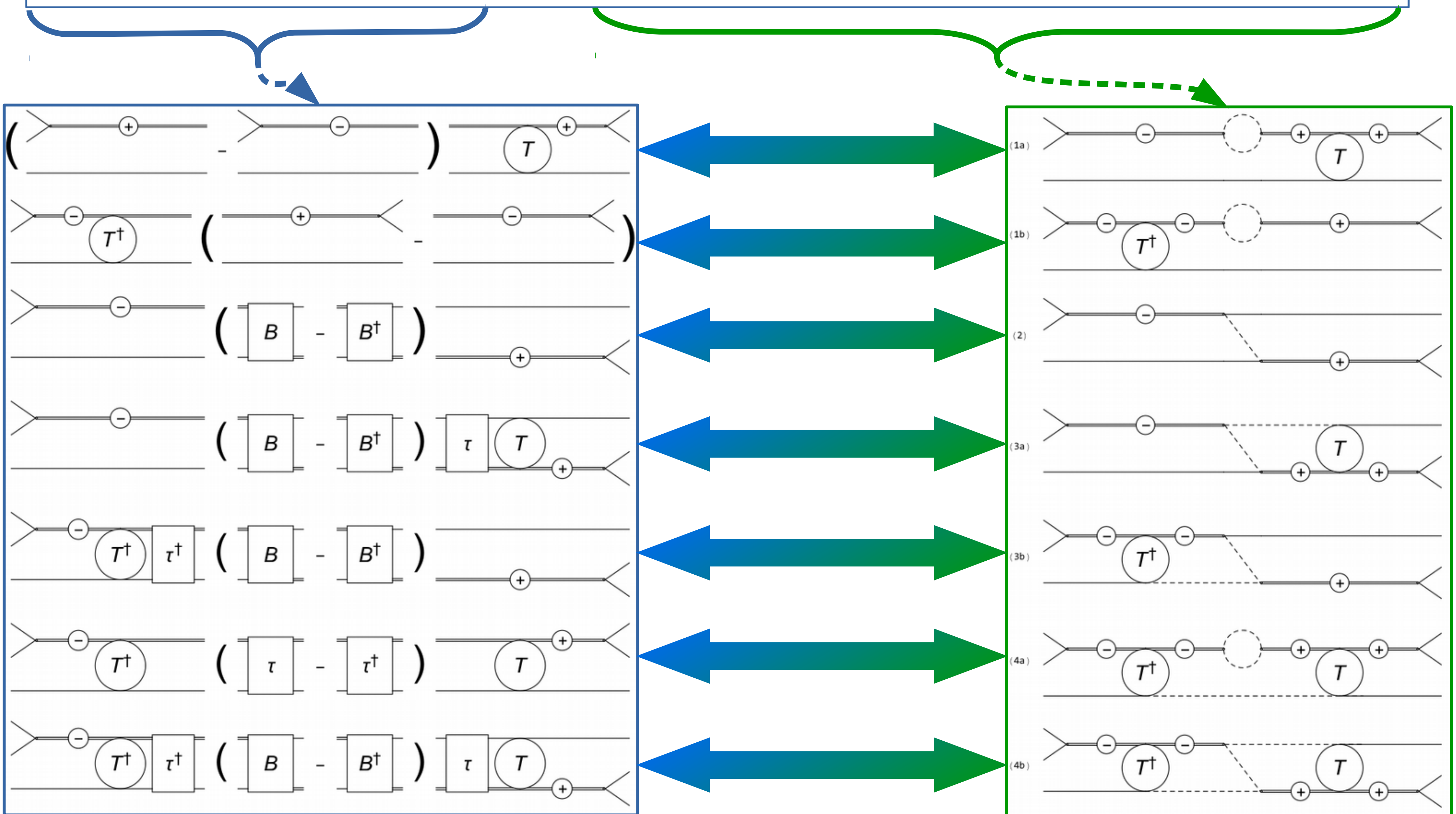
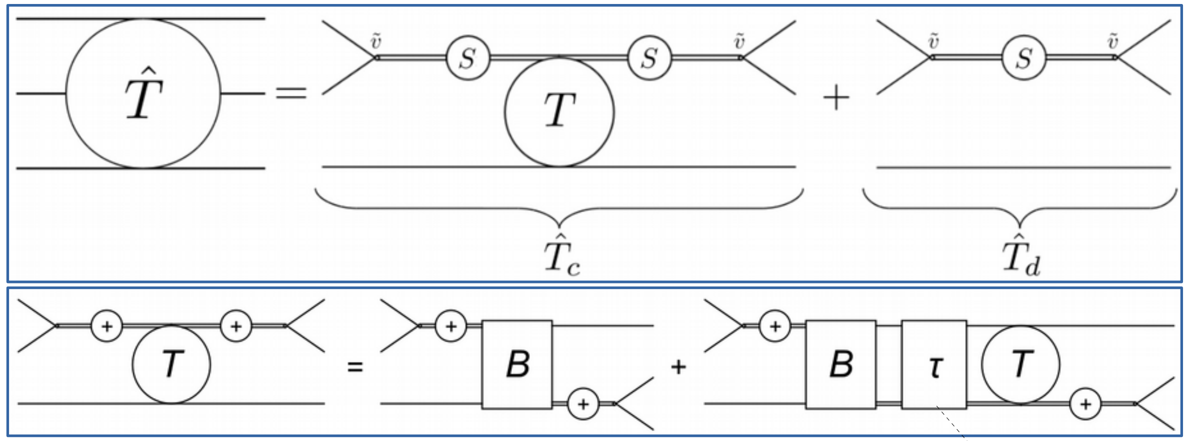


SCATTERING AMPLITUDE

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***υ=λ*** (full relations available)

$$\tau(\sigma(k)) = (2\pi)\delta^+(k^2 - m^2)S(\sigma(k))$$

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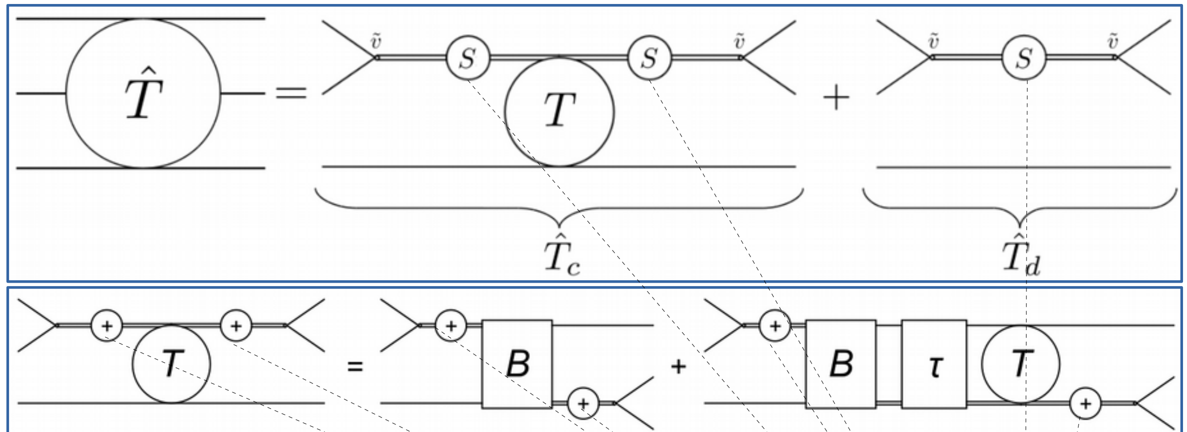

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$$\text{Disc}\,\frac{1}{S} = -\frac{i}{8\pi}\frac{K_{\rm cm}}{\sqrt{\sigma(k)}}\lambda^2$$

- twice subtracted dispersion relation in invariant mass - ***σ(k)***

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- in the rest-frame of isobar (***Lorentz invariance!***)

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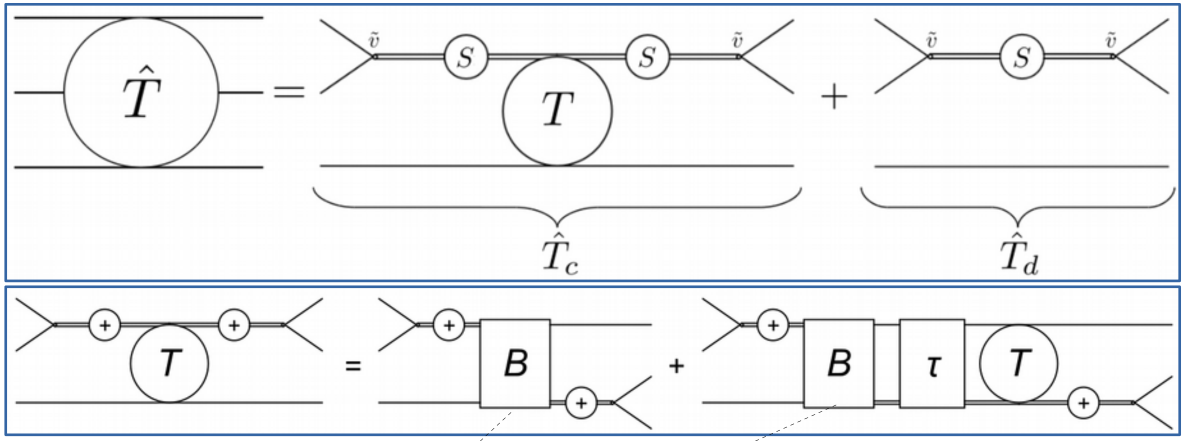

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$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

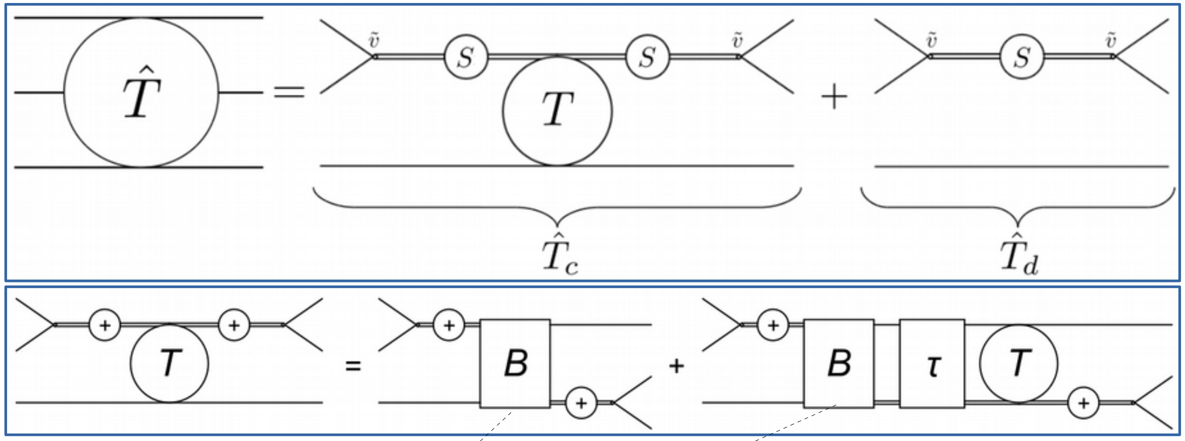
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$

- one-**π** exchange in TOPT → ***R E S U L T !***

SCATTERING AMPLITUDE

3 → 3 scattering amplitude is a 3-dimensional integral equation



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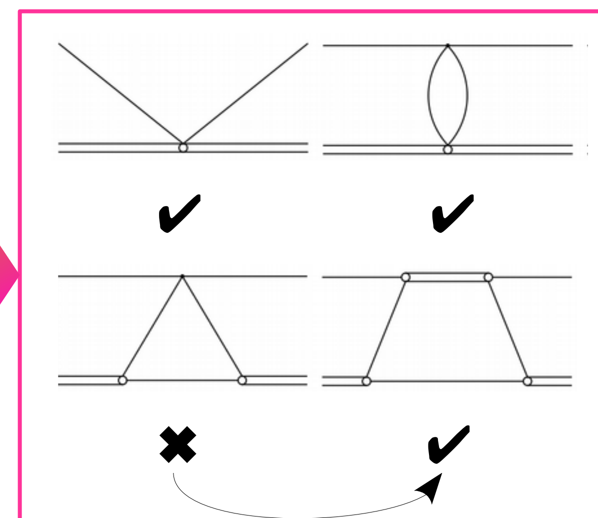
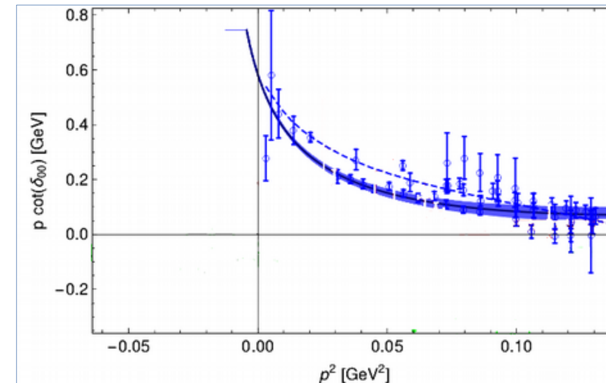


THREE-BODY AMPLITUDE IN A BOX

MM, Döring

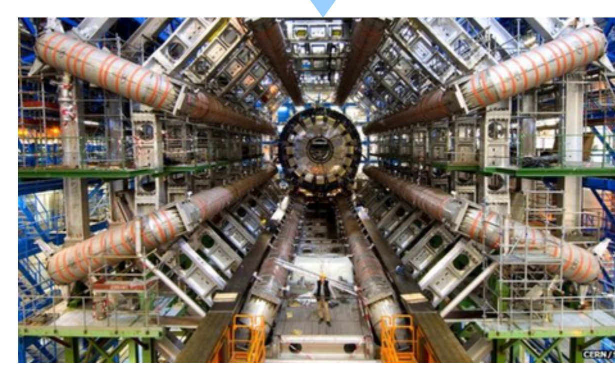
Arxiv: 1709.08222

WHY LATTICE?



ab-initio

D
A
T
A



Q C D

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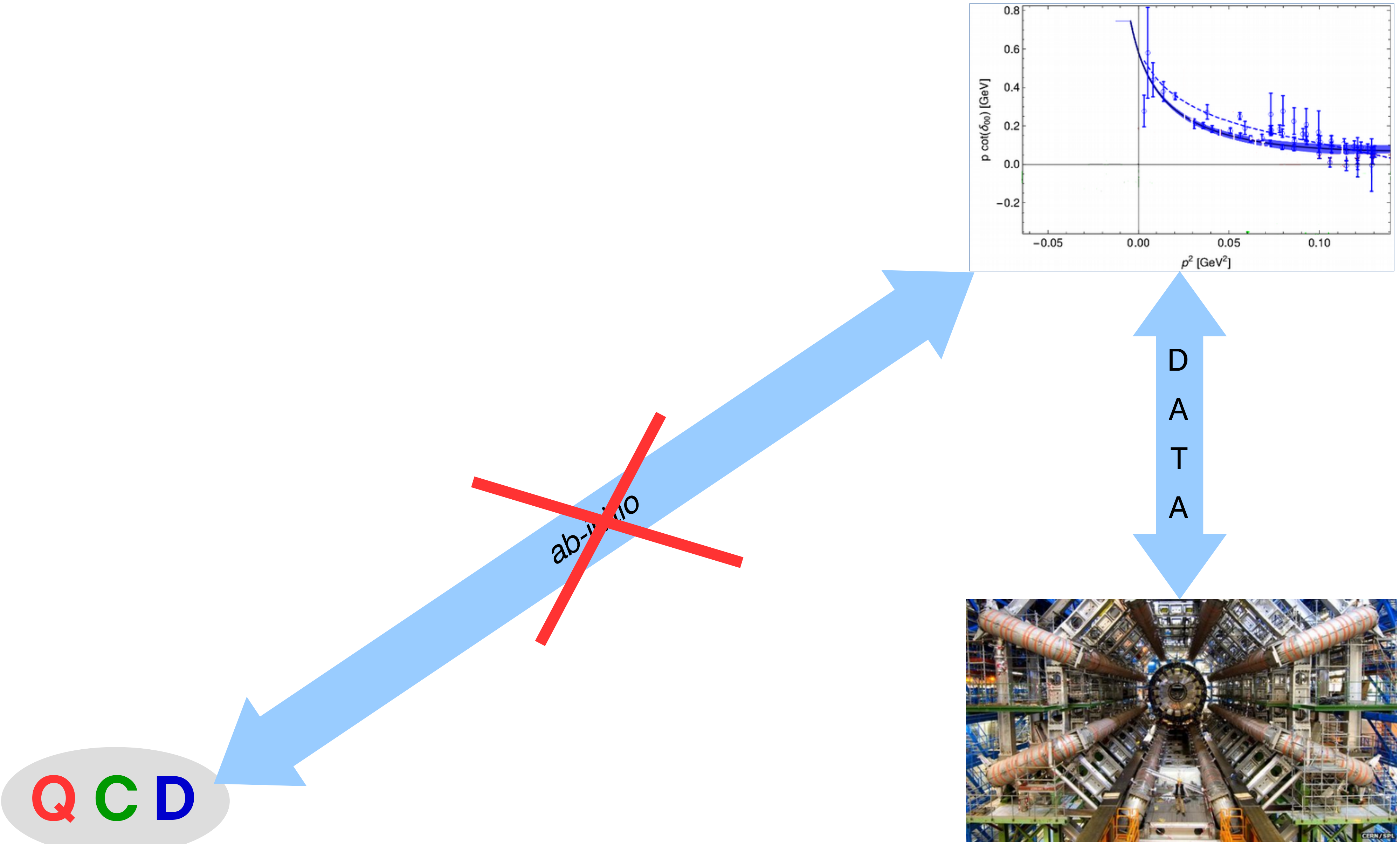

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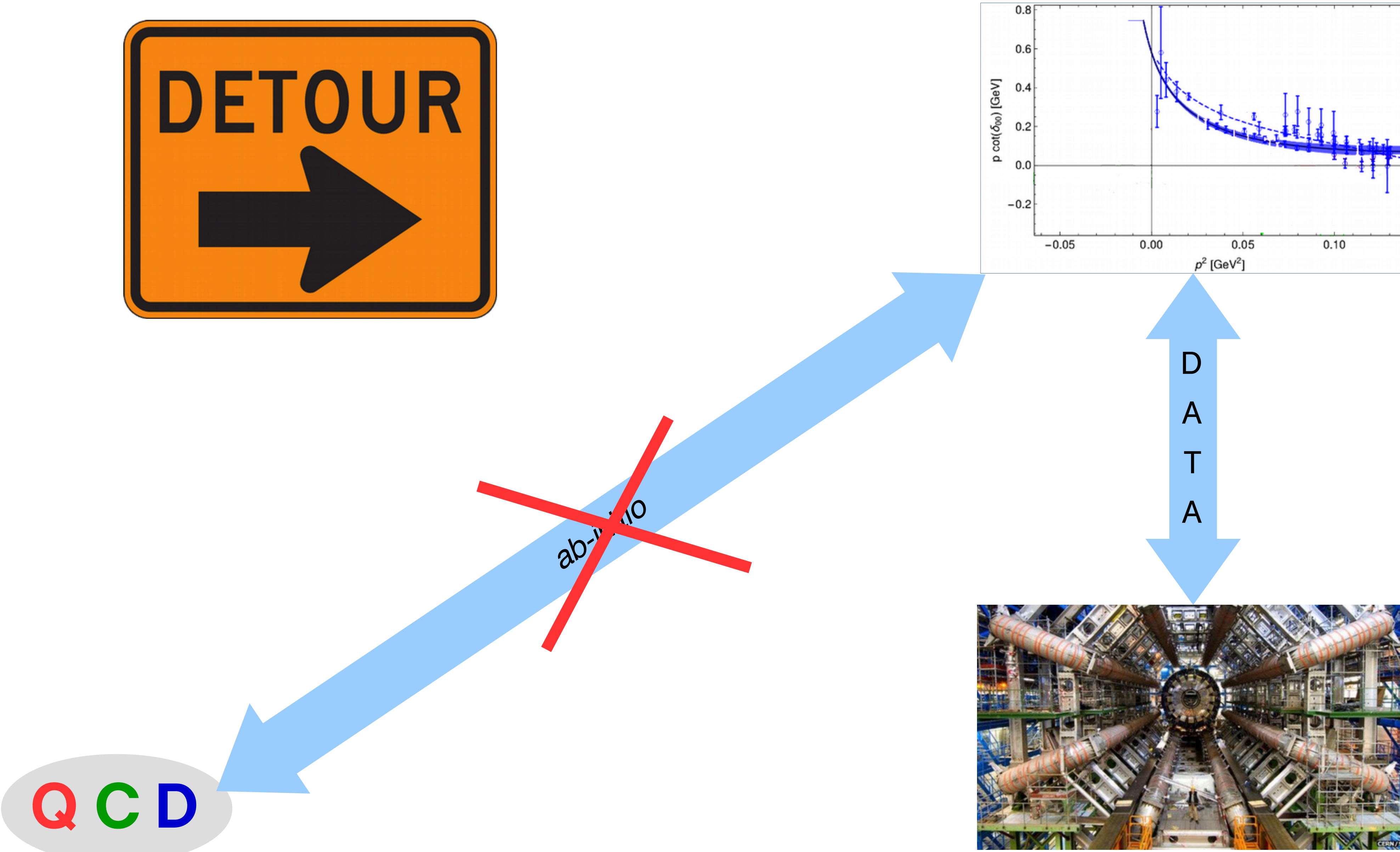

DETOUR
0.8
0.6
0.4
0.2
0.0
−0.2
−0.05
0.00
0.05
0.10
p cot(δ00) [GeV]
p² [GeV²]
ab-i
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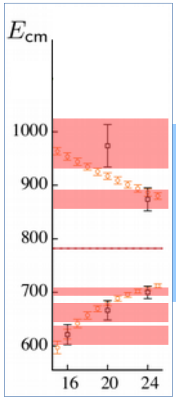
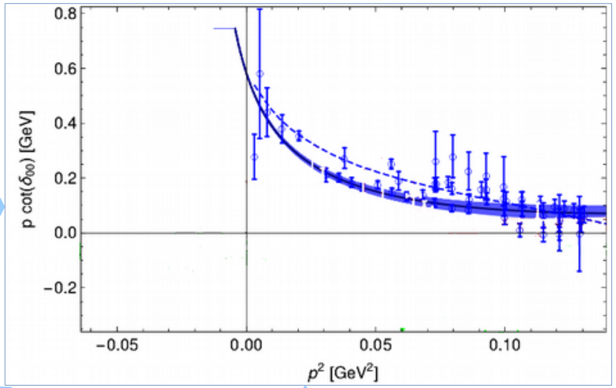


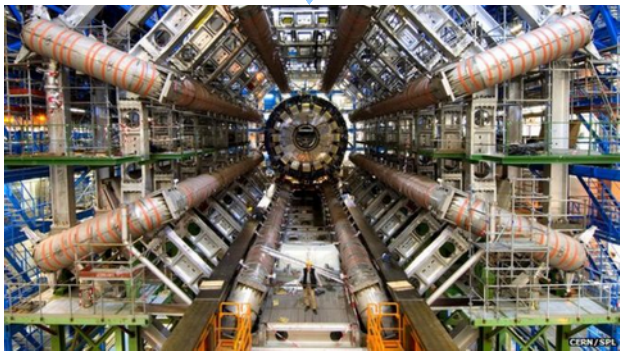
1) unphysical pion mass

2) (periodic) boundary conditions



ab-i io

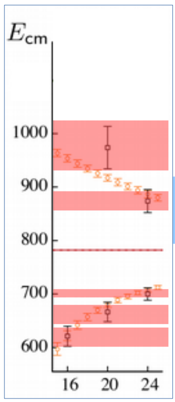
D A T A



Q C D

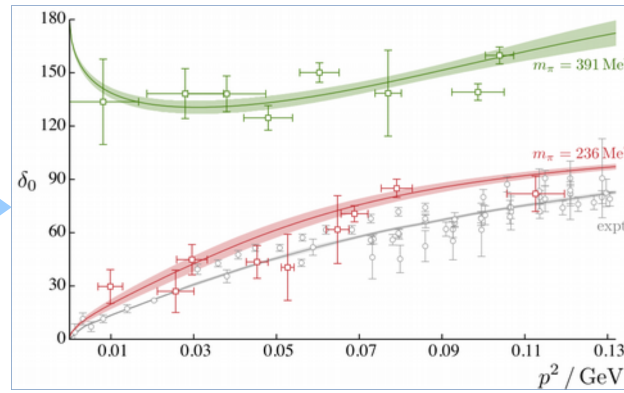
GOALS & CHALLENGES

Recipe for 2 → 2 scattering (e.g. *I=J=0* ππ scattering)



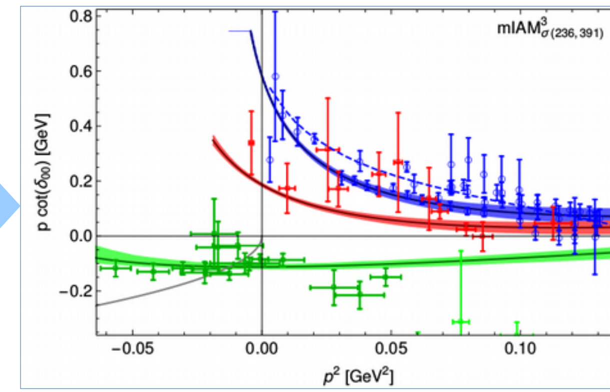
HSC(2016)

STEP 1



Briceño et al.(2016)

?



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- 1 eigenenergy ↔ 1 phase-shift in infinite volume
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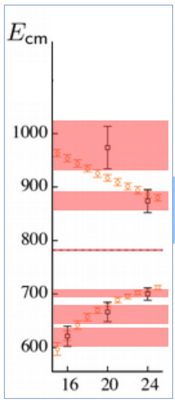

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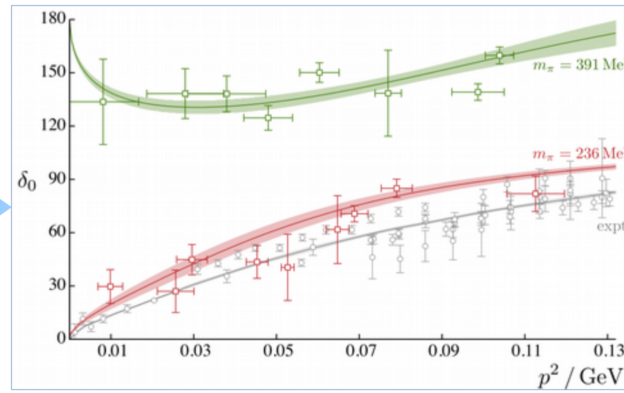

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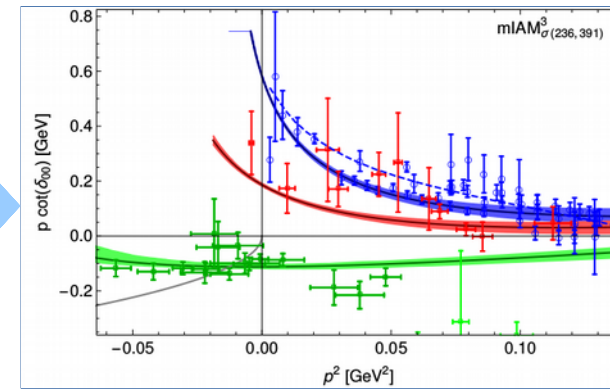

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CHIRAL EXTRAPOLATIONS

- M_π dependence from ChPT Gasser, Leutwyler(1981)
- Extensions to resonances exist

Hanhart et al. (2008)... Bruns, MM (2017)

GOALS & CHALLENGES

QCD calculations in finite volume

1) unphysical pion mass
2) (periodic) boundary conditions

→ **<u>discrete momenta</u>**

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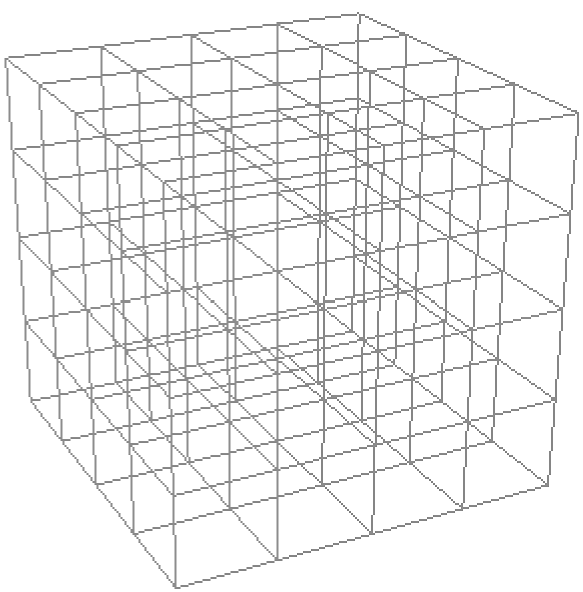

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→ <u>discrete momenta</u> & **<u>discrete spectrum</u>**

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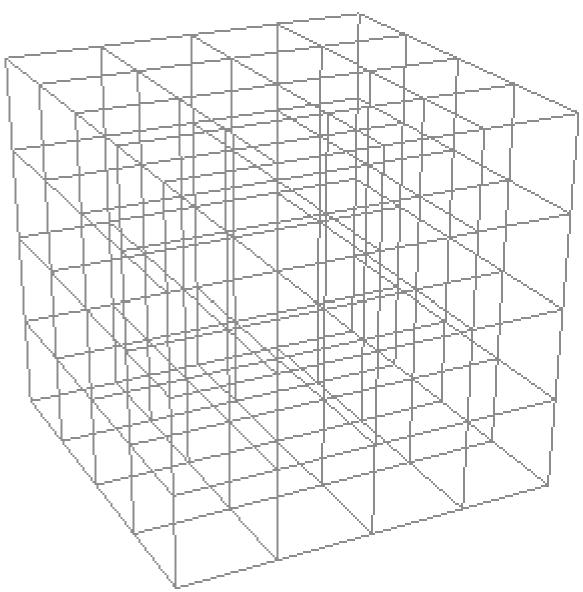

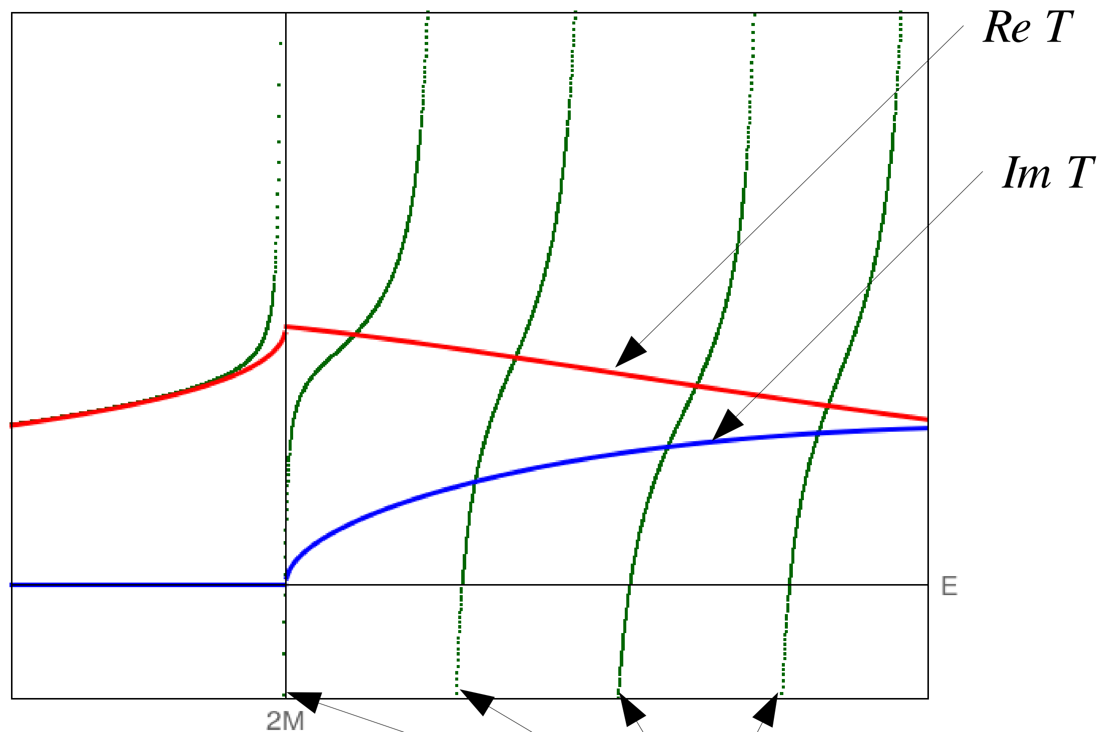

Energy eigenvalues ↔ *poles of* T^{FV}

GOALS & CHALLENGES

Lüscher-like formalism in $3 \to 3$ case is under investigation

Polejaeva / Rusetsky (2012) Briceño / Hansen / Sharpe (2016)

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⇒ THIS WORK: discretize 3 → 3 scattering amplitude in isobar formulation

GOAL: quantization condition from 3-body unitarity!

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Partial Waves in infinite volume

- separation of angular momentum → $Y_{lm}(\theta,\varphi)$
- reduces dimensionality of the problem

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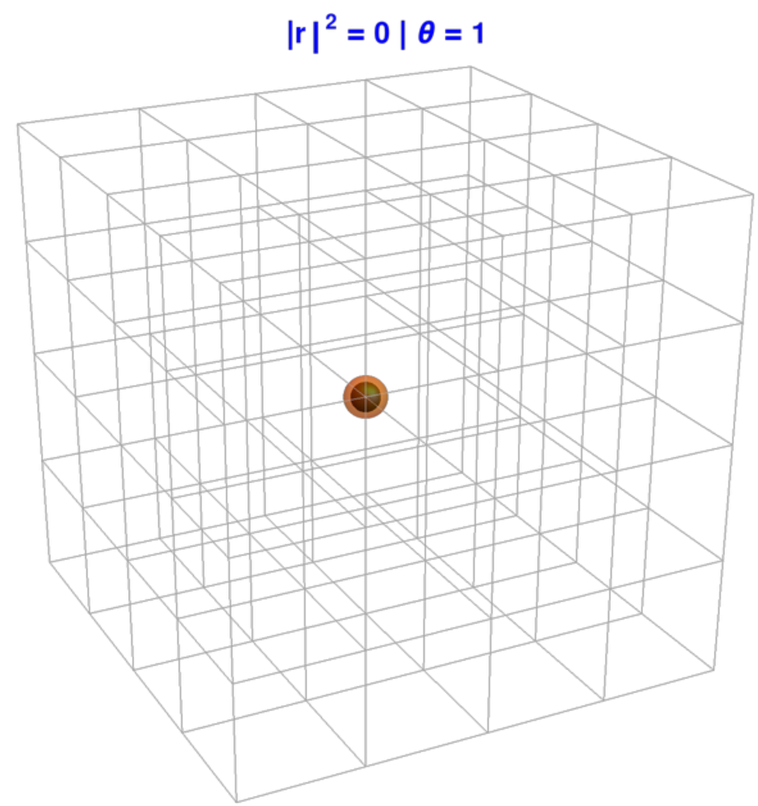

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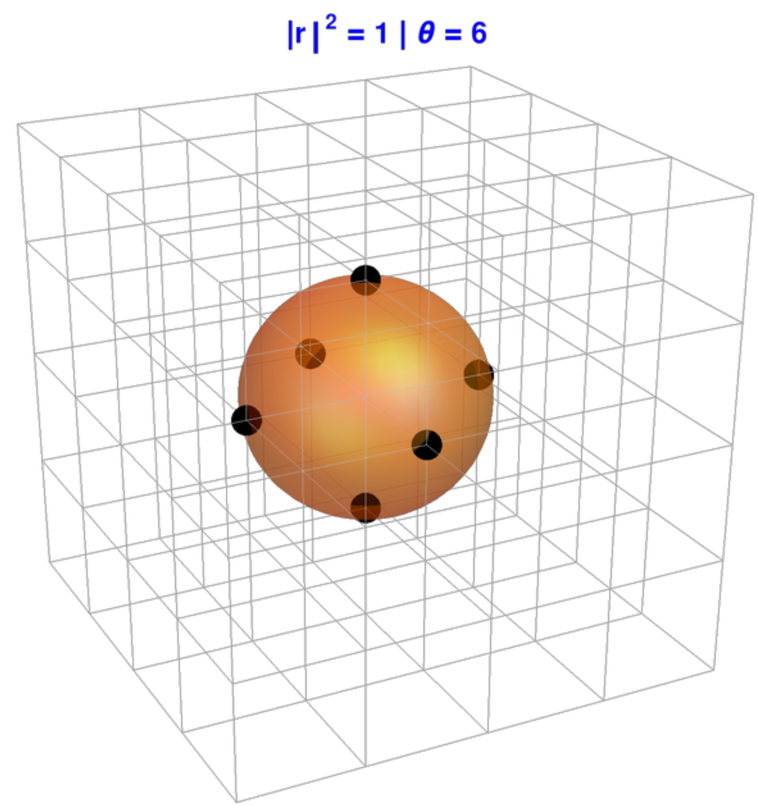

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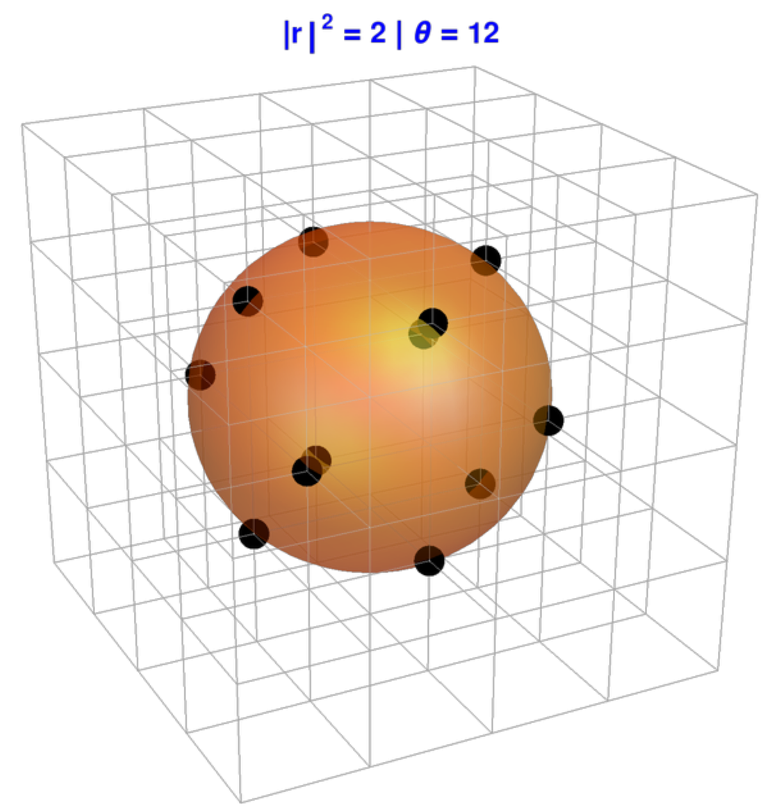

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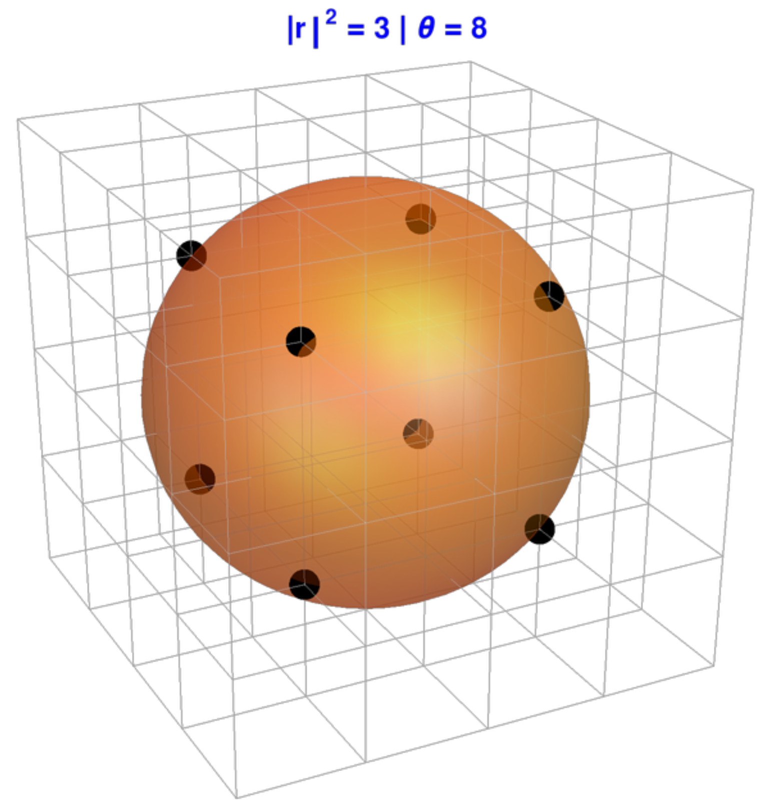

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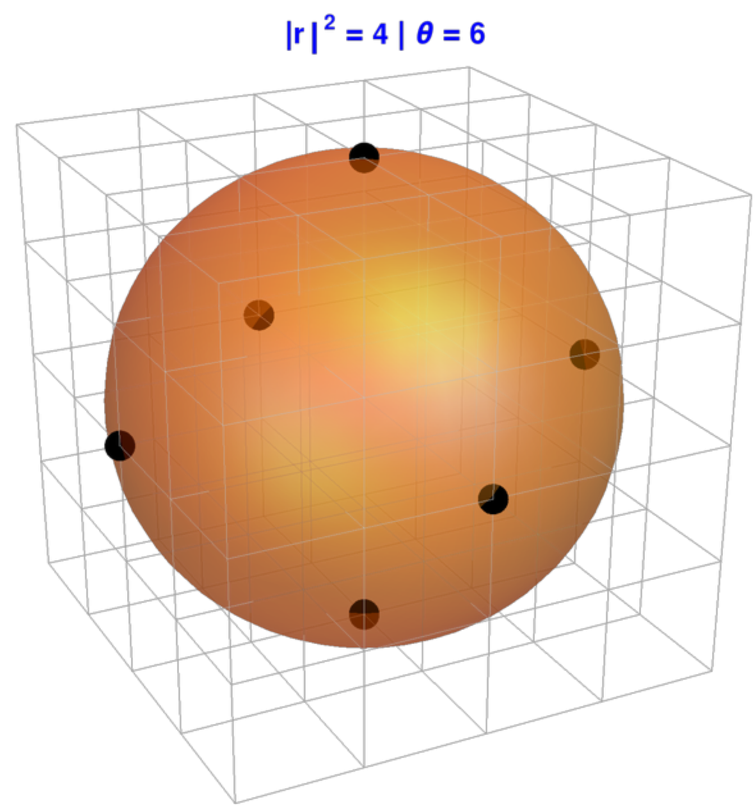

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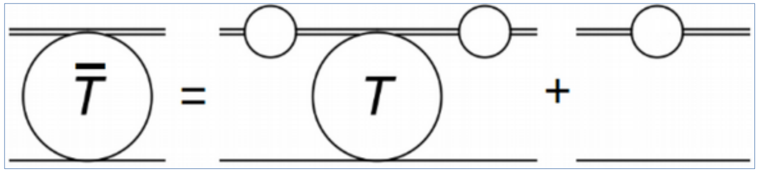
$\bar{T}(W)$ **is a matrix equation w.r.t** $|q|,\ |p|=0,1,2,3,4,5,6,8$

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s)T^{A_1^+}_{nm}(s)\tau_m(s) - 2E_n\tau_n(s)\frac{L^3}{\vartheta(n)}\delta_{nm}$$



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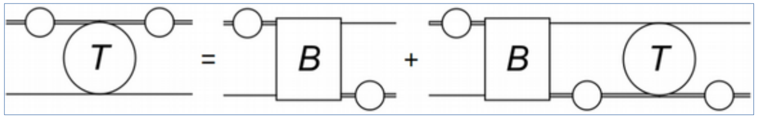

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τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\ell}\frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

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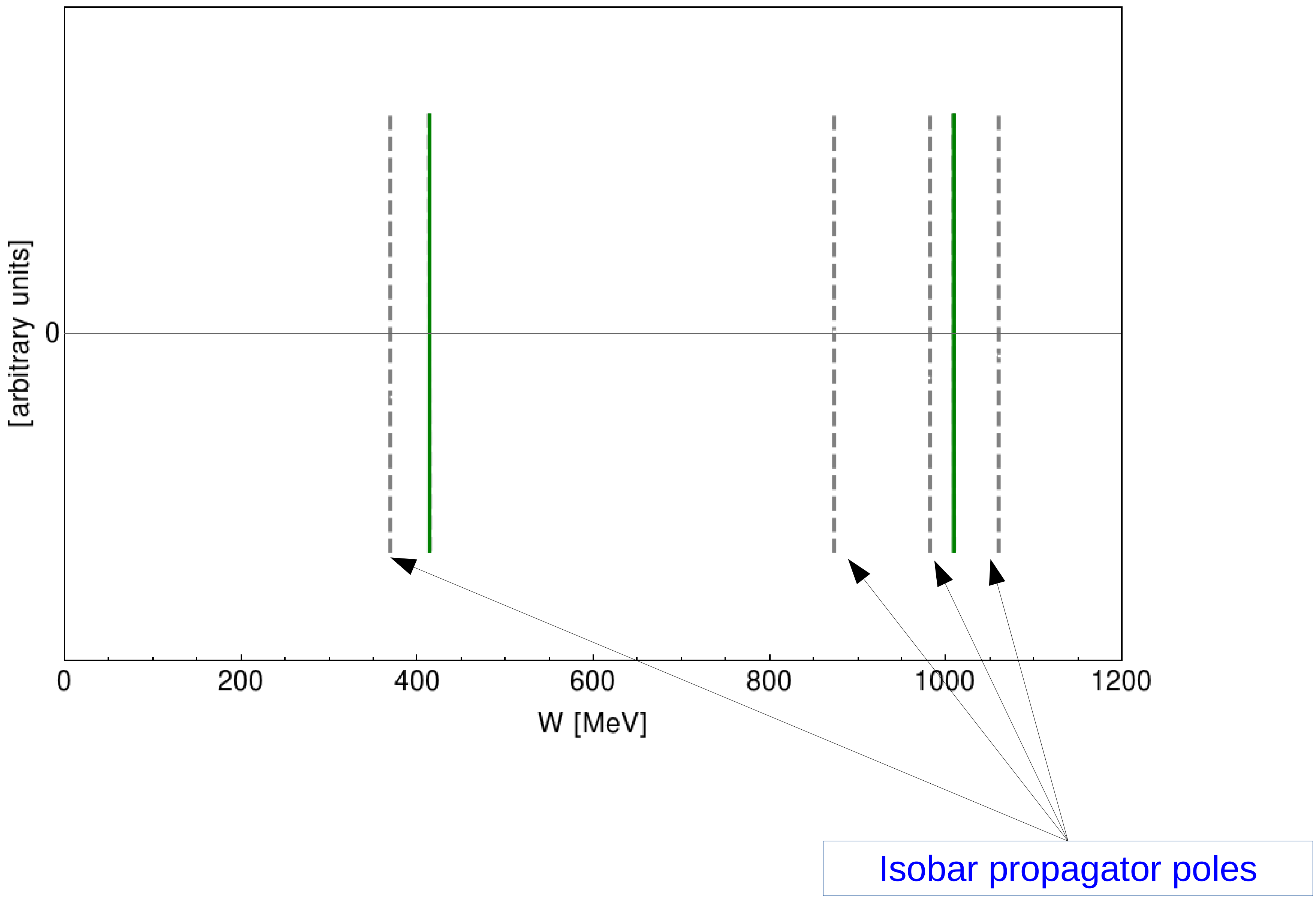
Genuine 3-body eigenenergies = poles in s:

$$\mathrm{Det}\left[B^{A_1^+}(s)\left[\frac{\vartheta(n)}{2E(s)L^3}\right] + \tau(s)^{-1}\right] = 0$$

RESULTS (L=3 fm, M=138 MeV)



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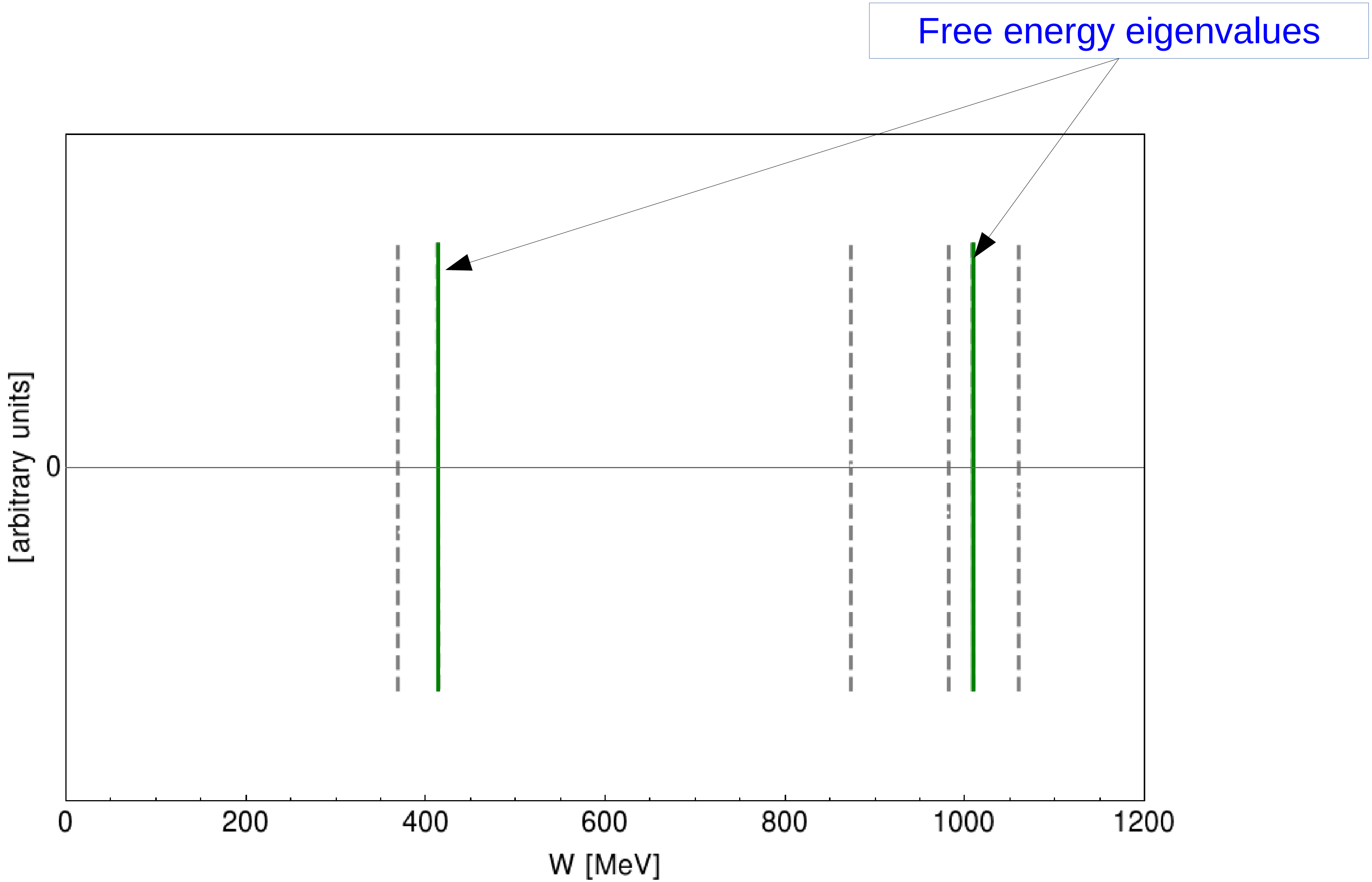

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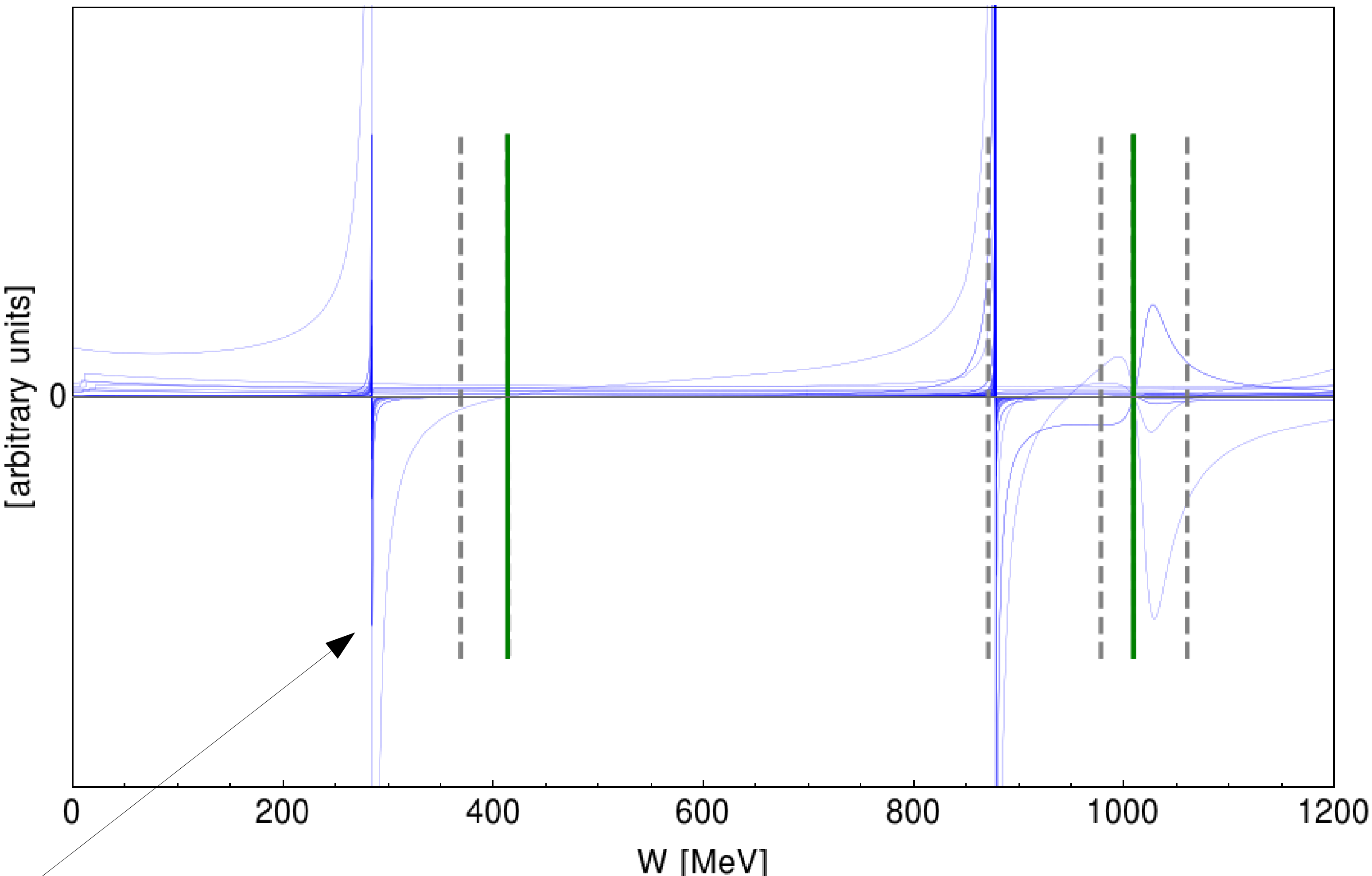

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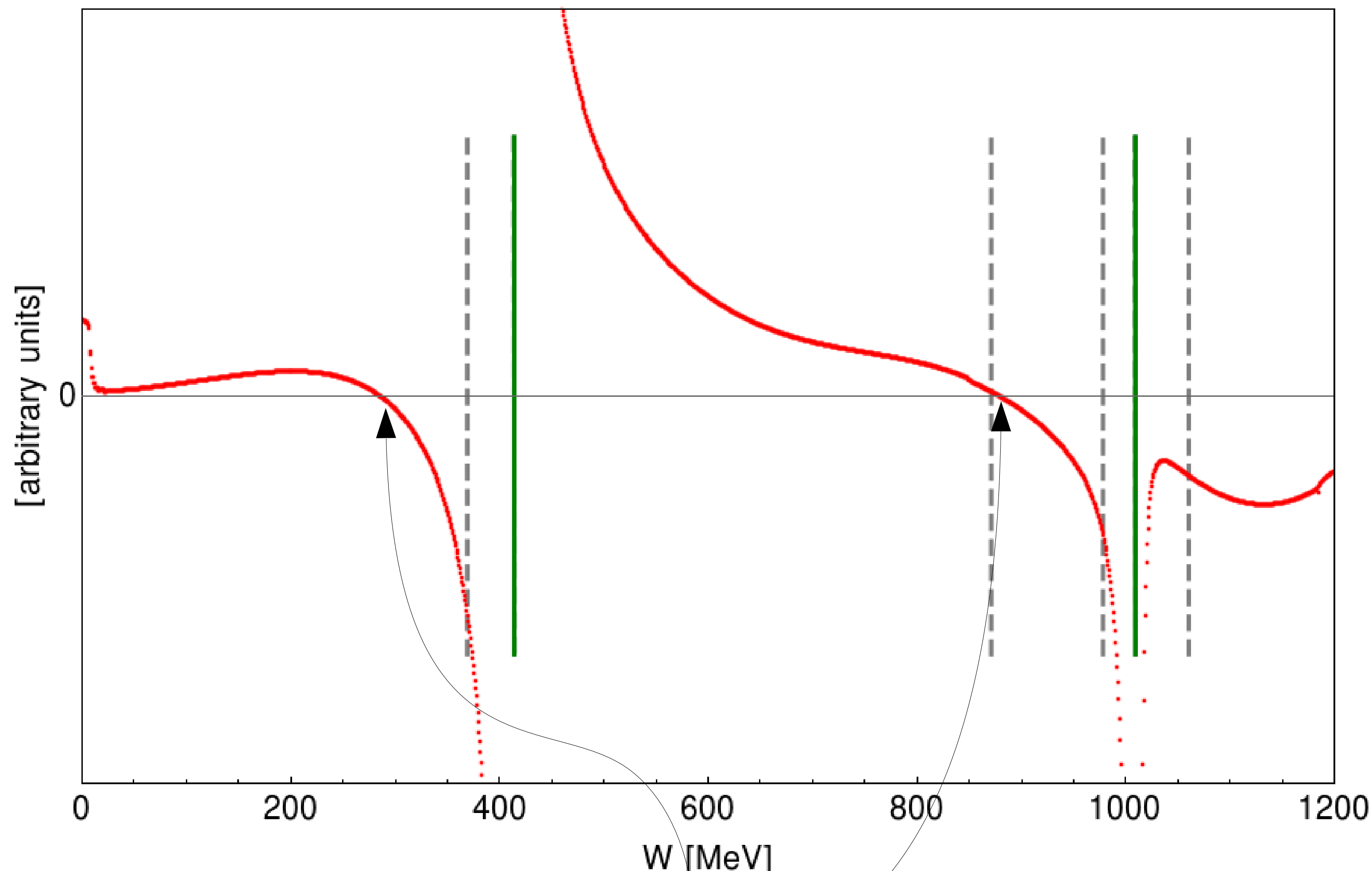

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SUMMARY

3-body amplitude in infinite volume

- 3-body Unitarity dictates imaginary parts of the driving term & isobar propagator
- Result: 3-dim. relativistic integral equations

Finite volume investigation:

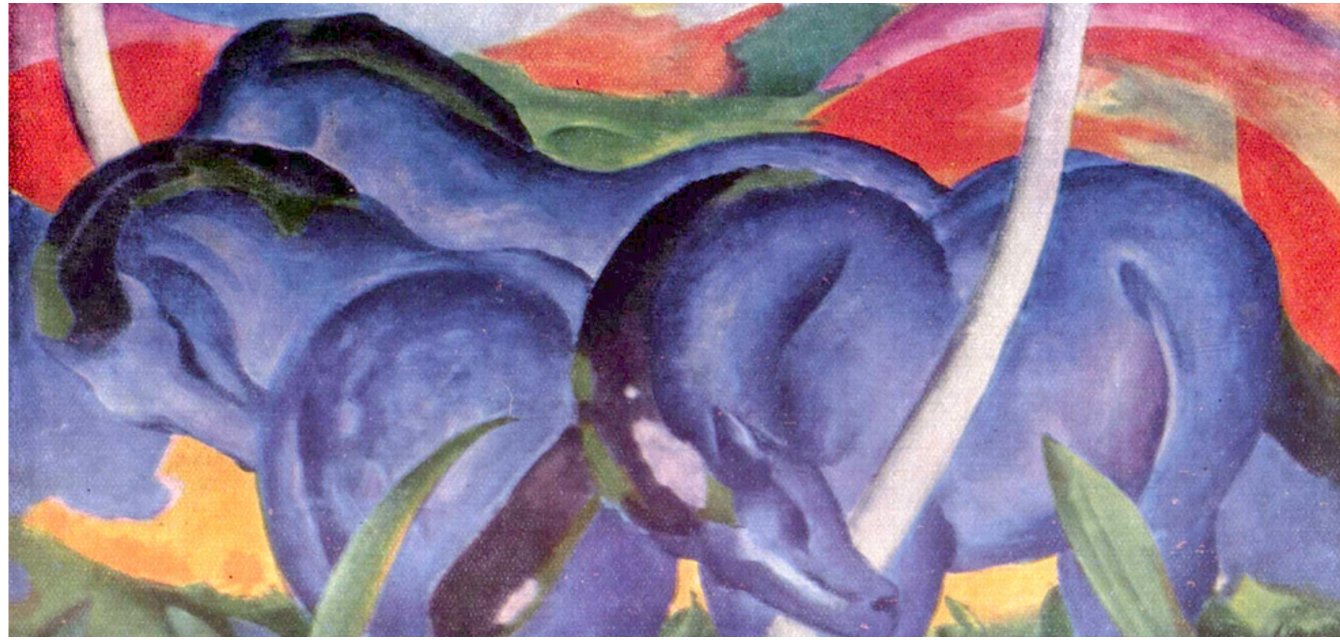
- Discretization techniques
- Quantization condition
- Case study → practicability!

OUTLOOK

→ include angular momentum / isospin / multiple isobars

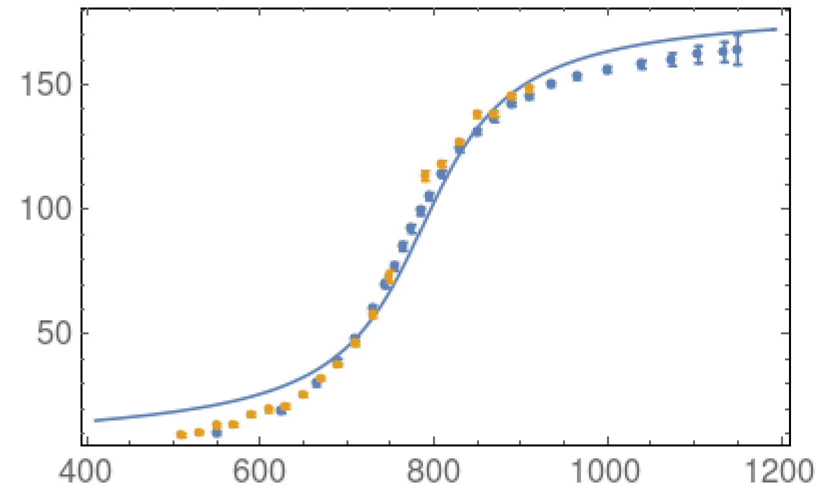
→ practical studies: a_1(1260), ...

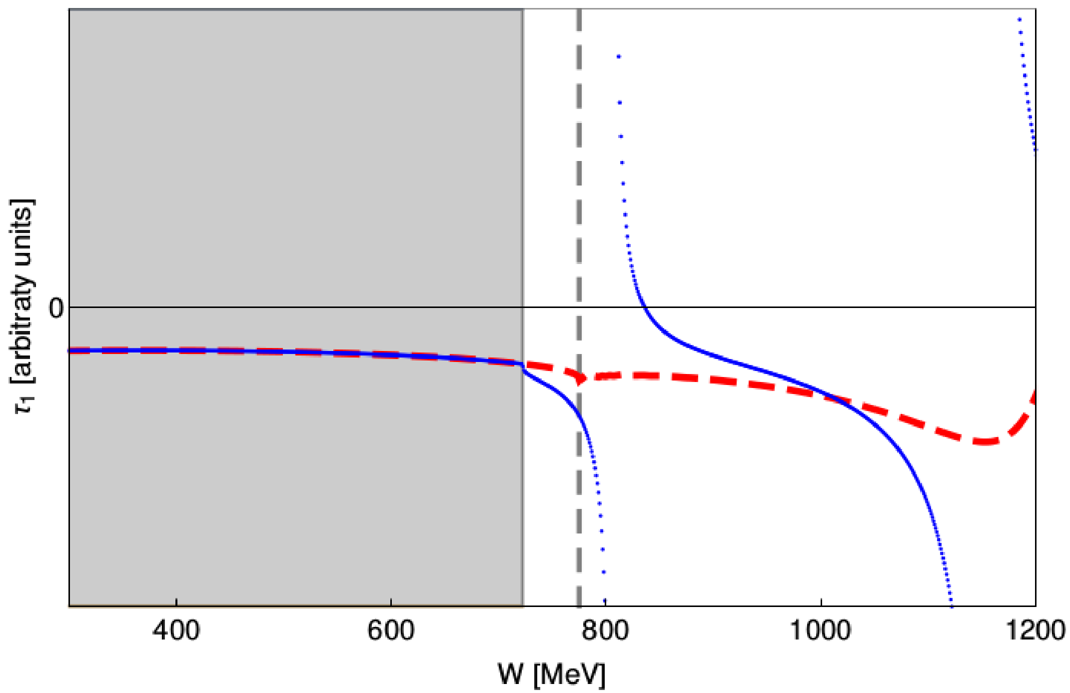
THANK YOU!



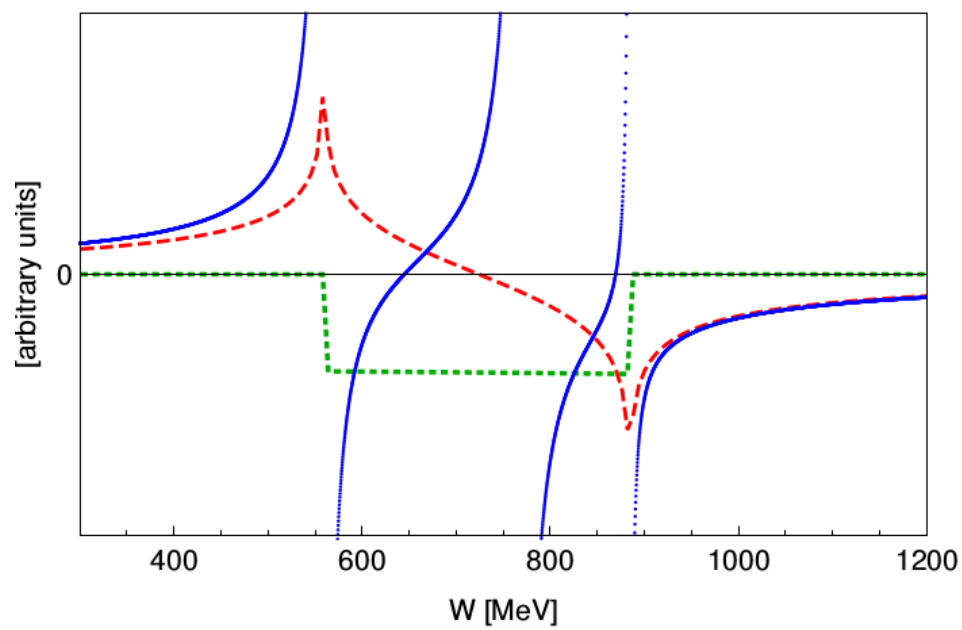
SPARES

- $T_{22}(W)$=v 1/D v
- 3 free parameter: β (form factor), λ (strength of coupling), M0 (“bare mass of isobar”)
- Fixed to reproduce typical phase-shifts
  → just to get into the same ballpark





τ_1 [arbitraty units]
0
400
600
800
1000
1200
W [MeV]



[arbitrary units]
0
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Unitarity & Matching

- 3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$