THE MUONIC HYDROGEN LAMB SHIFT AND THE PROTON RADIUS

Antonio Pineda

Universitat Autònoma de Barcelona

Institute of theoretical physics II, Bochum, May 3rd, 2013

Precise measurements in atomic physics → Learning about hadron structure

Hyperfine splitting (hydrogen atom):

$$E_{HF}^{exp} = E(n=1, s=1) - E(n=1, s=0) \qquad (s = \text{total spin})$$

Nature (1972)

$$\nu_{HF} = \frac{E_{HF}}{h} = 1420.4057517667(9)\ \text{MHz} \qquad (13\ \text{digits})$$

Precise measurements in atomic physics → Learning about hadron structure

Hyperfine splitting (hydrogen atom):

$$E_{HF}^{exp} = E(n=1, s=1) - E(n=1, s=0) \qquad (s = \text{total spin})$$

Nature (1972)

$$\nu_{HF} = \frac{E_{HF}}{h} = 1420.4057517667(9) \text{ MHz} \qquad (13 \text{ digits})$$

Precise measurements in atomic physics $\rightarrow$ Learning about hadron structure

Hyperfine splitting (hydrogen atom):

$$E_{HF}^{exp} = E(n=1, s=1) - E(n=1, s=0) \qquad (s = \text{total spin})$$

Nature (1972)

$$\nu_{HF} = \frac{E_{HF}}{h} = 1420.4057517667(9)\ \text{MHz} \qquad (13\ \text{digits})$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262\frac{r_p^2}{fm^2} + 0.0347\frac{r_p^3}{fm^3}\ meV = 205.984\ meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311\ meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23)\ {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51)\ {\rm meV}$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262 \frac{r_p^2}{fm^2} + 0.0347 \frac{r_p^3}{fm^3} \; meV = 205.984 \; meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311 \; meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23) \; {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51) \; {\rm meV}$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262 \frac{r_p^2}{fm^2} + 0.0347 \frac{r_p^3}{fm^3} \; meV = 205.984 \; meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311 \; meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23) \; {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51) \; {\rm meV}$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262 \frac{r_p^2}{fm^2} + 0.0347 \frac{r_p^3}{fm^3}\ meV = 205.984\ meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311\ meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23)\ {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51)\ {\rm meV}$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262\frac{r_p^2}{fm^2} + 0.0347\frac{r_p^3}{fm^3}\ meV = 205.984\ meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311\ meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23)\ {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51)\ {\rm meV}$$

Precise measurements in atomic physics → Learning about hadron structure
Lamb shift (muonic hydrogen)

$$E \equiv E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$$

PSI: R. Pohl et al., Nature vol. 466, p. 213 (2010)

$$E_{exp} = 206.2949(32) meV$$

$$E_{th} = 209.9779(49) - 5.2262 \frac{r_p^2}{fm^2} + 0.0347 \frac{r_p^3}{fm^3} \; meV = 205.984 \; meV$$

using CODATA value $r_p = 0.8768(69)/0.8775(51)$ fm.

$$E_{exp} - E_{th} = 0.311 \; meV$$

New proposed value: $r_p = 0.84184(67)/0.84087(39)$ fm. 5/7 standard deviations!!

A. Antognini et al., Science vol. 339, p. 417 (2013)

$$\Delta E_L^{\rm exp} = 202.3706(23) \; {\rm meV}$$

$$\Delta E_{HF}^{\rm exp} = 22.8089(51) \; {\rm meV}$$

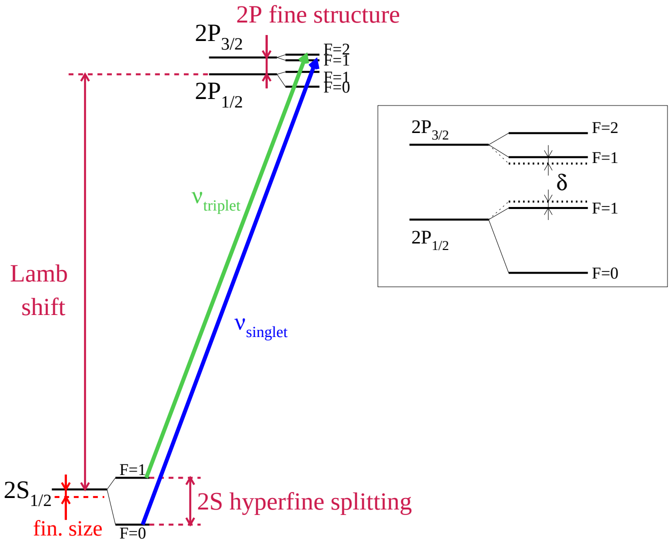


Figure : *From 1208.2637. A. Antognini et al..*

Theoretical setup

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r}\right)\psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons} \end{array} \right\} \text{potential NRQED} \qquad E \sim mv^2$$

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$

$m_r \alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r\alpha}{m_r} \sim \frac{m_r\alpha^2}{m_r\alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r\alpha^5$ (heavy quarkonium precision)

Theoretical setup

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} \right) \psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons} \end{array} \right\} \text{potential NRQED} \qquad E \sim mv^2$$

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$

$m_r \alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r \alpha}{m_r} \sim \frac{m_r \alpha^2}{m_r \alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r \alpha^5$ (heavy quarkonium precision)

Theoretical setup

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu\alpha \gg m_\mu\alpha^2$

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r}\right)\psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons}\end{array}\right\}\text{potential NRQED} \qquad E \sim mv^2$$

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p+m_\mu}$

$m_r\alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r\alpha}{m_r} \sim \frac{m_r\alpha^2}{m_r\alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r\alpha^5$ (heavy quarkonium precision)

Theoretical setup

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} \right) \psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons} \end{array} \right\} \text{potential NRQED} \qquad E \sim mv^2$$

Scales:

$m_p \sim \Lambda_\chi$

$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$

$m_r \alpha \sim m_e$

Expansion parameters, ratios between scales, mainly:

$$\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \sim \frac{1}{9}$$

$$\frac{m_r \alpha}{m_r} \sim \frac{m_r \alpha^2}{m_r \alpha} \sim \alpha \sim \frac{1}{137}$$

Needed precision $m_r \alpha^5$ (heavy quarkonium precision)

Theoretical setup

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r},\mathbf{R},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$
$$-V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)\Big\} S(\mathbf{r},\mathbf{R},t) - \int d^3\mathbf{r}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k)\frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha\frac{1}{1+\Pi(-\mathbf{k}^2)}\,,$$

where

$$\Pi(k^2) = \alpha\Pi^{(1)}(k^2) + \alpha^2\Pi^{(2)}(k^2) + \alpha^3\Pi^{(3)}(k^2) + \dots$$

$$\alpha_V(k) = \alpha_{eff}(k) + \sum_{\substack{n,m=0\\ n+m=even>0}} Z_\mu^n Z_p^m \alpha_{eff}^{(n,m)}(k) = \alpha_{eff}(k) + \delta\alpha(k)\,, \qquad \delta\alpha(k) = O(\alpha^4)$$

Theoretical setup

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r},\mathbf{R},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)\Big\} S(\mathbf{r},\mathbf{R},t) - \int d^3\mathbf{r}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k)\frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha\frac{1}{1+\Pi(-\mathbf{k}^2)}\,,$$

where

$$\Pi(k^2) = \alpha\Pi^{(1)}(k^2) + \alpha^2\Pi^{(2)}(k^2) + \alpha^3\Pi^{(3)}(k^2) + \dots$$

$$\alpha_V(k) = \alpha_{eff}(k) + \sum_{\substack{n,m=0\\ n+m=even>0}} Z_\mu^n Z_p^m \alpha_{eff}^{(n,m)}(k) = \alpha_{eff}(k) + \delta\alpha(k)\,, \qquad \delta\alpha(k) = O(\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$

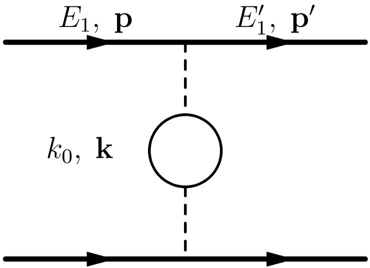
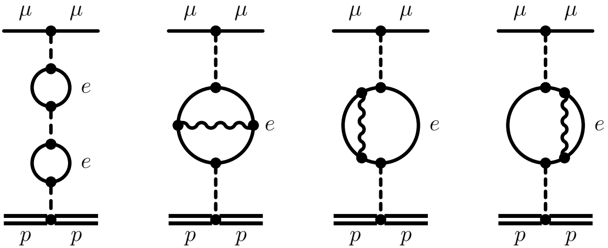


Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

1-loop static potential

$$E_{LO} = \langle n|\delta V|n\rangle = 205.0074\ \mathrm{meV} = \mathcal{O}(m_r\alpha^3)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



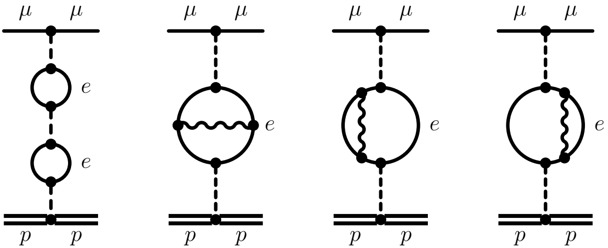
Pachuki/Borie
2-loop static potential is the same as two-loop vacuum polarization iterations
(*two loop vacuum polarization*)

$$\delta E = \langle n|\delta V|n\rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n|\delta V \frac{1}{H_C - E_n}\delta V|n\rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



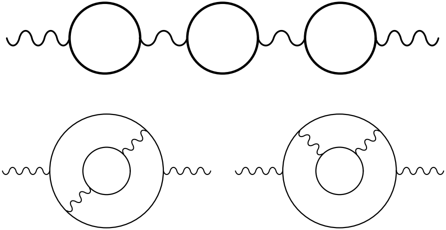
Pachuki/Borie
2-loop static potential is the same as two-loop vacuum polarization iterations (*two loop vacuum polarization*)

$$\delta E = \langle n|\delta V|n\rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n|\delta V\frac{1}{H_C - E_n}\delta V|n\rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^5)$

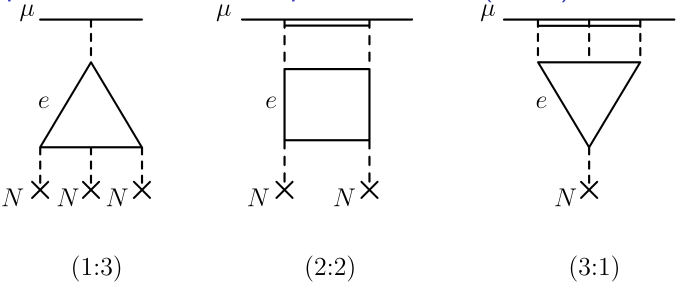


3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

$$0.0076\ \mathrm{meV} = \mathcal{O}(m_r\alpha^5)$$

Slightly corrected by Ivanov et al.

Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



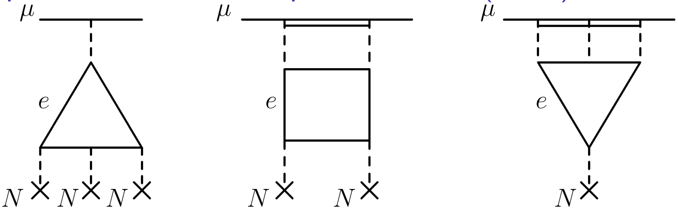
Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ mev} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

Observation:
The limit $m_e \to 0$ known from QCD (Anzai et al. and Smirnov et al).
It should be possible to obtain the result with finite mass (albeit numerically) and check.

Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



(1:3) (2:2) (3:1)

Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ mev} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

Observation:
The limit $m_e \to 0$ known from QCD (Anzai et al. and Smirnov et al).
It should be possible to obtain the result with finite mass (albeit numerically) and check.

$1/m$ potential

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(1)}(r)}{m_\mu} \to \mathcal{O}(m_r\alpha^6)$$

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} ,$$

$$V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r \alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia)

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X}, t)\right\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \rightarrow \mathcal{O}(m_r\alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim, Ivanov, Korzinin)

Matching NRQED to pNRQED. Getting the potential

relativistic corrections+vacuum polarization

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x}, \mathbf{X}, t) \Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$- V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{x} \cdot \mathbf{E}(\mathbf{X}, t) \Big\} S(\mathbf{x}, \mathbf{X}, t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

$$V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r \alpha^4, \alpha^5)$$

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim, Ivanov, Korzinin)

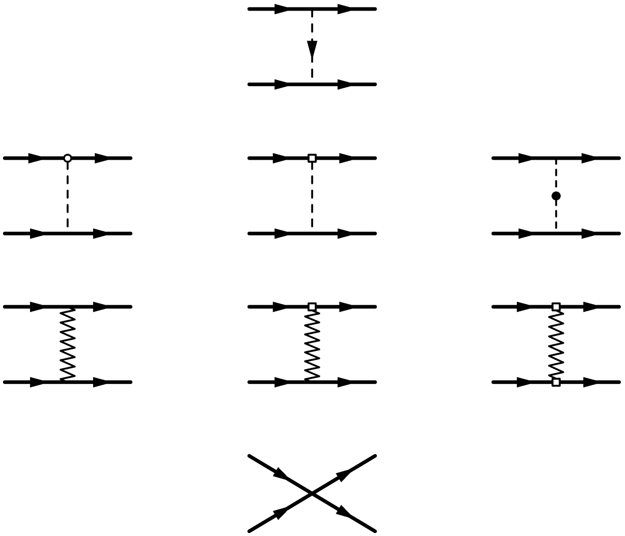
Matching NRQED to pNRQED. Getting the potential

$$\mathcal{L}_\mu = \mu^\dagger \Bigg\{ iD_0 + c_k \frac{\mathbf{D}^2}{2m} + c_4 \frac{\mathbf{D}^4}{8m^3} + c_F\, g \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2m} + c_D\, g \frac{(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} + i c_S\, g \frac{\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} \Bigg\} \mu,$$

$$\mathcal{L}_p = N_p^\dagger \Bigg\{ iD_0 + c_k \frac{\mathbf{D}^2}{2m_p} + c_4 \frac{\mathbf{D}^4}{8m_p^3} + c_F\, g \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2m_p} + c_D\, g \frac{(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m_p^2} + i c_S\, g \frac{\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m_p^2} \Bigg\} N_p,$$

and

$$\mathcal{L}_{\mu p} = \frac{c_3}{m^2} \mu^\dagger \mu N_p^\dagger N_p + \frac{c_4}{m^2} \mu^\dagger \boldsymbol{\sigma} \mu N_p^\dagger \boldsymbol{\sigma} N_p \,.$$



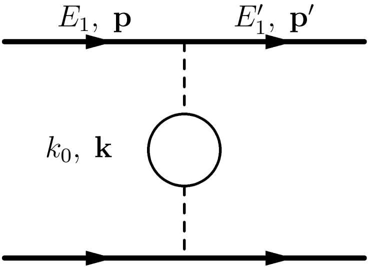


Figure : *Typical one-loop correction to the previous diagrams coming from the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha_{eff}(k)}{2}\left[Z_p\frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu\frac{c_D^{(p)}}{m_p^2}\right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha_{eff}(k)\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left\{Z_p\frac{c_S^{(\mu)}\mathbf{s}_1}{m_\mu^2} + Z_\mu\frac{c_S^{(p)}\mathbf{s}_2}{m_p^2}\right\},$$

$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha\left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2}\right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p\frac{4\pi\alpha_{eff}(k)}{m_\mu m_p}\left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^4}\right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha_{eff}(k)}{m_\mu m_p}\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left(Z_p c_F^{(\mu)}\mathbf{s}_1 + Z_\mu c_F^{(p)}\mathbf{s}_2\right),$$

$$\tilde{V}^{(g)} = \frac{4\pi\alpha_{eff}(k)c_F^{(1)}c_F^{(2)}}{m_\mu m_p}\left(\mathbf{s}_1\cdot\mathbf{s}_2 - \frac{\mathbf{s}_1\cdot\mathbf{k}\mathbf{s}_2\cdot\mathbf{k}}{\mathbf{k}^2}\right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2}\left\{(c_{3,NR}^{pl_i} + 3c_{4,NR}^{pl_i}) - 2c_{4,NR}^{pl_i}\mathbf{S}^2\right\}.$$

Order $1/m^2$ from energy-dependent terms

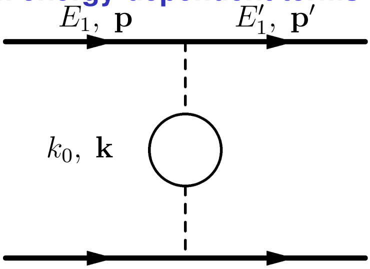


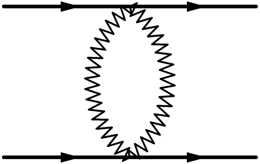
Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

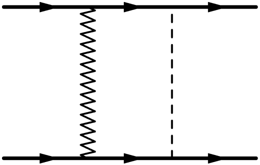
$$\delta \tilde{V}_E = -\frac{Z_\mu Z_p e^2}{4 m_\mu m_p} \frac{(\mathbf{p}^2 - \mathbf{p}'^2)^2}{\mathbf{k}^2} \frac{\alpha}{\pi} m_e^2 \int_4^\infty d(q^2) \frac{1}{(m_e^2 q^2 + \mathbf{k}^2)^2} u(q^2) .$$

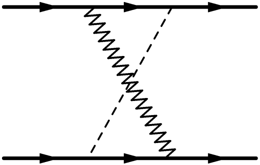
$$u(q^2) = \frac{1}{3}\sqrt{1 - \frac{4}{q^2}}\left(1 + \frac{2}{q^2}\right) .$$

$$\delta V_E + V^{(e)}\Big|_{\rm 1loop} = -\frac{Z_\mu Z_p \alpha^2}{\pi}\frac{m_e^2}{m_\mu m_p}\int_4^\infty d(q^2)\frac{u(q^2)}{(m_e q)^2}$$
$$\left\{\frac{1}{2}\left\{\mathbf{p}^2, \frac{e^{-m_e q r}}{r}\left(1+\frac{m_e q r}{2}\right)\right\}\right.$$
$$\left. -\frac{e^{-m_e q r}}{2r^3}\left(1+m_e q r\right)\mathbf{L}^2 + \frac{(m_e q)^2}{4r}e^{-m_e q r}\left(1+\frac{m_e q r}{2}\right) - 2\pi\delta(\mathbf{r})\right\},$$

where **L** is the angular momentum. It agrees with the corresponding expression by Pachucki.

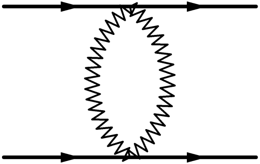


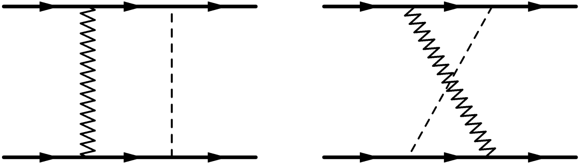




$$\tilde{V}^{(a)}_{1\,loop} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3}\log 2 + \frac{5}{3}\right),$$

$$\tilde{V}^{(b,c)}_{1\,loop} = \frac{4 Z_\mu^2 Z_p^2 \alpha^2}{3 m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} + 2\log 2 - 1\right).$$

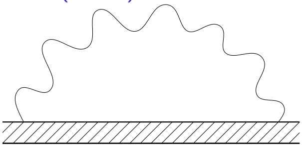




$$\tilde{V}^{(a)}_{1\,loop} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3}\log 2 + \frac{5}{3}\right)\,,$$

$$\tilde{V}^{(b,c)}_{1\,loop} = \frac{4 Z_\mu^2 Z_p^2 \alpha^2}{3 m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} + 2\log 2 - 1\right)\,.$$

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



Pachucki

$$\Delta E = -0.6677\ meV$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}): \qquad \Delta E = -0.045\ meV$$

Start the overlap with hadronic effects.

Hadronic corrections

$$L_{pNRQED} = \int d^3x d^3X dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E} N_p + \cdots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t) \Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$- V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t) \Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \ldots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E} N_p + \cdots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2}c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p\mu^\dagger\mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2}c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger \mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \rightarrow \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2}c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \dots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \dots + \frac{c_3}{m_p^2}N_p^\dagger N_p\mu^\dagger\mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2}c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p\mu^\dagger\mu$$

Hadronic corrections

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}\delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.}(m_r\alpha)^3$$

$$D_d^{had.} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2}c_D$$

c_3, d_2, c_D, ... matching coefficients of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger\mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U D^\mu U\right] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p\mu^\dagger\mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4} \mathrm{Tr}\left[D_\mu U D^\mu U\right] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu \nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla} \cdot \mathbf{E} N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla} \cdot \mathbf{E} N_p + \cdots + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\,[D_\mu U D^\mu U] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger\mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\,[D_\mu U D^\mu U] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger\mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\,[D_\mu U D^\mu U] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger \mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U D^\mu U\right] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots\frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p + \cdots + \frac{c_3}{m_p^2}N_p^\dagger N_p \mu^\dagger\mu$$

Hadronic vacuum polarization effects

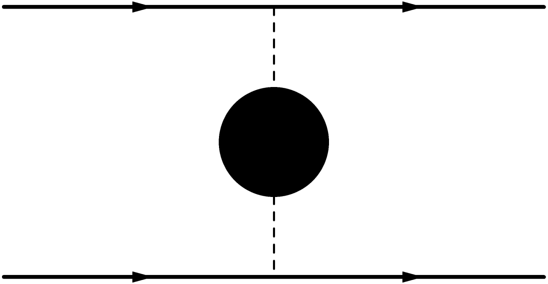


Figure : *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \to$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

Hadronic vacuum polarization effects

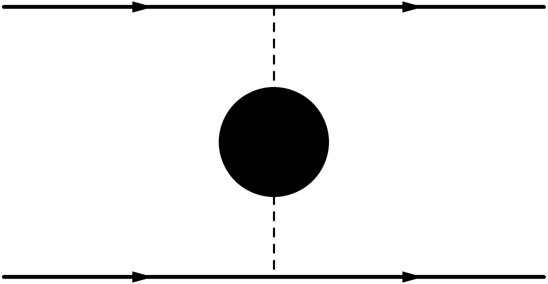


Figure : *Leading correction to the Coulomb potential due to the hadronic vacuum polarization.*

$d_2 \to$ hadronic vacuum polarization

$$\Delta E = 0.011 \text{ meV}$$

c_3 or Zemach (r^3) effects: $\mathcal{O}(m_\mu\alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order)
m_μ extra suppression

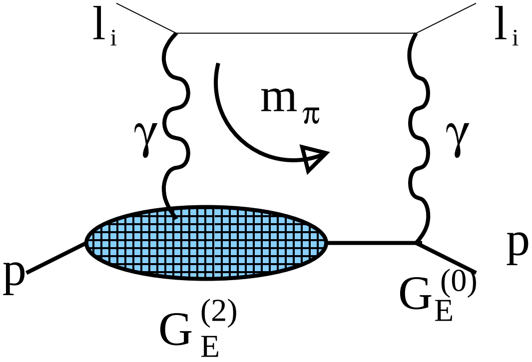


Figure : *Symbolic representation (plus permutations) of the Zemach $\langle r^3 \rangle$ correction.*

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\mathrm{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\mathrm{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\delta c^{pl_i}_{3,Zemach} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r_p^3 \rangle = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0}\right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g^2_{\pi N\Delta} \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta}\right)^{2r} + g^2_{\pi N\Delta} \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta}\right)^{2r} \right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \qquad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln\left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\mathrm{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\mathrm{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\delta c_{3,Zemach}^{pl_i} = \frac{\pi}{3}\alpha^2 m_p^2 m_\mu \langle r_p^3 \rangle = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0}\right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta}\right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta}\right)^{2r} \right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \qquad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln\left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\mathrm{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\mathrm{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\frac{\langle r_p^3 \rangle|_{\chi PT}}{\mathrm{fm}^3} = 1.9 \text{ (Pineda)} \rightarrow \Delta E = 0.019 \text{ meV}$$

$$\frac{\langle r_p^3 \rangle|_{\text{"}exp\text{"}}}{\mathrm{fm}^3} = \left\{ \begin{array}{l} 2.71(13) \text{ Friar - Sick} \\ 2.50 \text{ Arrington} \\ 2.85(8) \text{ Bernauer - Arrington} \end{array} \right\} \rightarrow \Delta E = 0.025 - 0.029$$

Not the reason for the discrepancy.

$\langle r_p^3 \rangle \sim 35$ De Rujula, not consistent neither with experiment nor chiral symmetry.

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\mathrm{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\mathrm{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\frac{\langle r_p^3 \rangle|_{\chi PT}}{\mathrm{fm}^3} = 1.9 \ (\mathrm{Pineda}) \to \Delta E = 0.019 \ \mathrm{meV}$$

$$\frac{\langle r_p^3 \rangle|_{''exp''}}{\mathrm{fm}^3} = \left\{ \begin{array}{l} 2.71(13) \ \mathrm{Friar-Sick} \\ 2.50 \ \mathrm{Arrington} \\ 2.85(8) \ \mathrm{Bernauer-Arrington} \end{array} \right\} \to \Delta E = 0.025 - 0.029$$

Not the reason for the discrepancy.

$\langle r_p^3 \rangle \sim 35$ De Rujula, not consistent neither with experiment nor chiral symmetry.

$$\Delta E = 0.010 \frac{\langle r_p^3 \rangle}{\mathrm{fm}^3}$$

$$\frac{\langle r_p^3 \rangle}{\mathrm{fm}^3} = \frac{96}{\pi} \int d^{D-1}k \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

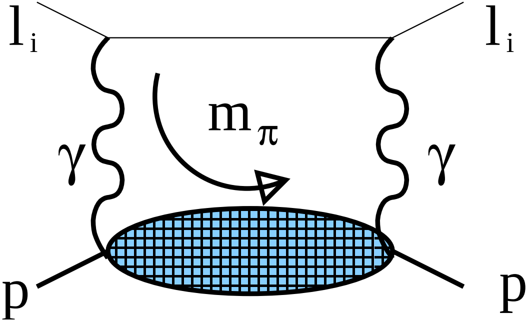
$$\frac{\langle r_p^3 \rangle|_{\chi PT}}{\mathrm{fm}^3} = 1.9 \text{ (Pineda)} \to \Delta E = 0.019 \text{ meV}$$

$$\frac{\langle r_p^3 \rangle|_{\text{"}exp\text{"}}}{\mathrm{fm}^3} = \left\{ \begin{array}{l} 2.71(13) \text{ Friar} - \text{Sick} \\ 2.50 \text{ Arrington} \\ 2.85(8) \text{ Bernauer} - \text{Arrington} \end{array} \right\} \to \Delta E = 0.025 - 0.029$$

Not the reason for the discrepancy.

$\langle r_p^3 \rangle \sim 35$ De Rujula, not consistent neither with experiment nor chiral symmetry.

c_3 Polarizability effects: $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$



$$c_{3,NR}^{pl_i} = -e^4 m_p m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4 m_{l_i}^2 k_{0,E}^2}$$
$$\times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) S_1(ik_{0,E}, -k_E^2) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned} T^{\mu\nu} &= \left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2) \\ &+\frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2)\end{aligned}$$

$\Delta E(\textit{Dispersion relations} + \textit{Model}) = 0.012(\text{Pachucki})/0.015(\text{Borie})\ \text{meV}$

Can one do better? $\rightarrow \chi$PT (Model independent)

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order)

m_μ extra suppression

$$\mathcal{O}(m_\mu\alpha^5\times\frac{m_\mu^2}{\Lambda_\chi^2}\times\frac{m_\mu}{m_\pi})$$

$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned} T^{\mu\nu} \quad &= \quad \left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2) \\ &\quad + \frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2) \end{aligned}$$

$\Delta E(\textit{Dispersion relations} + \textit{Model}) = 0.012(\mathrm{Pachucki})/0.015(\mathrm{Borie})\ \mathrm{meV}$
Can one do better? $\rightarrow \chi$PT (Model independent)
Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order)
m_μ extra suppression

$$\mathcal{O}(m_\mu\alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

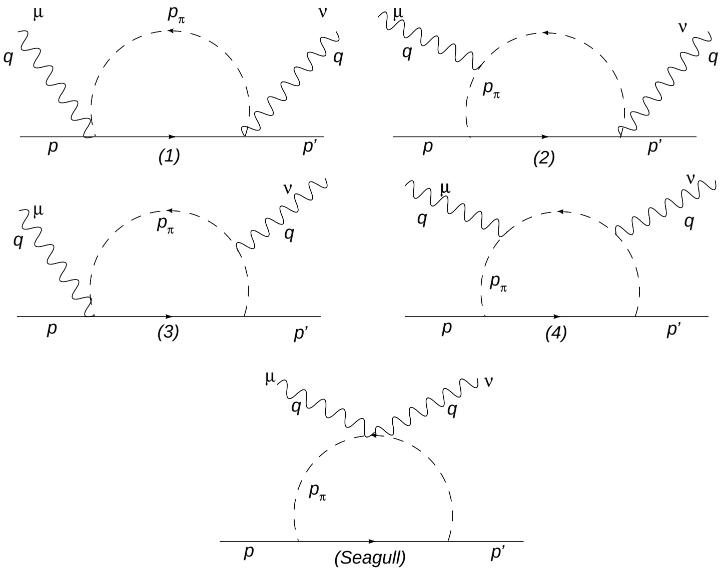


Figure : Diagrams contributing to T^{ij}. Crossed diagrams are not explicitly shown but calculated.

$$c_{3,NR}^{pl_i} = -e^4 m_p^2 \frac{m_{l_i}}{m_\pi}\left(\frac{g_A}{f_\pi}\right)^2 \int \frac{d^{D-1}k_E}{(2\pi)^{D-1}} \frac{1}{(1+\mathbf{k}^2)^4}$$
$$\times \int_0^\infty \frac{dw}{\pi} w^{D-5} \frac{1}{w^2 + 4\frac{m_{l_i}^2}{m_\pi^2}\frac{1}{(1+\mathbf{k}^2)^2}}$$
$$\times \left\{(2+(1+\mathbf{k}^2)^2)A_E(w^2,\mathbf{k}^2) + (1+\mathbf{k}^2)^2\mathbf{k}^2 w^2 B_E(w^2,\mathbf{k}^2)\right\}$$

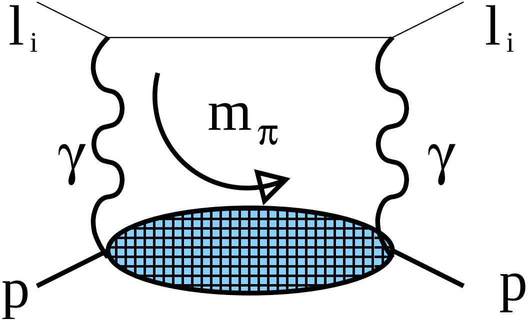
where (for $D=4$)

$$A_E = -\frac{1}{4\pi}\left[-\frac{3}{2} + \sqrt{1+w^2} + \int_0^1 dx \frac{1-x}{\sqrt{1+x^2w^2+x(1-x)w^2\mathbf{k}^2}}\right],$$

$$B_E = \frac{1}{8\pi}\left[\int_0^1 dx \frac{1-2x}{\sqrt{1+x^2w^2+x(1-x)w^2\mathbf{k}^2}} - \frac{1}{2}\int_0^1 dx \frac{(1-x)(1-2x)^2}{(1+x^2w^2+x(1-x)w^2\mathbf{k}^2)^{\frac{3}{2}}}\right].$$

c_3 Polarizability effects: $\mathcal{O}(m_\mu\alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order)
m_μ extra suppression

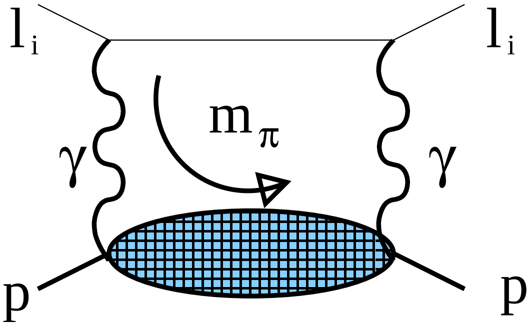


$\Delta E|(\textit{Dispersion relations} + \textit{Model}) = 0.012(\text{Pachucki})/0.015(\text{Borie})$ meV

$\Delta E|_{\chi PT}(\textit{pions}) = 0.018(\text{Nevado} - \text{Pineda})$ meV Model independent!!

c_3 Polarizability effects: $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$

Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order)
m_μ extra suppression



$$\Delta E|(\textit{Dispersion relations} + \textit{Model}) = 0.012(\text{Pachucki})/0.015(\text{Borie})\ \text{meV}$$

$$\Delta E|_{\chi PT}(\textit{pions}) = 0.018(\text{Nevado} - \text{Pineda})\ \text{meV} \quad \text{Model independent!!}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + ...$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + ...$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$$

Definition of the proton radius

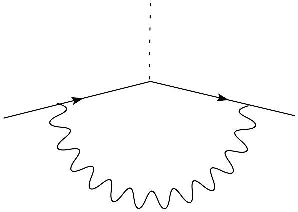
$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$

Infrared divergent! $\rightarrow$ Wilson coefficient



Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0}\,,$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

$$c_{D,point-like} = 1 + \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\frac{m_p^2}{\nu^2}\right)$$

$$\Delta E = -5.19745 * 0.8768^2 \simeq -4\ \mathrm{meV}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0}\,,$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

$$c_{D,point-like} = 1 + \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\frac{m_p^2}{\nu^2}\right)$$

$$\Delta E = -5.19745 * 0.8768^2 \simeq -4\ \mathrm{meV}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0}\,,$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

$$c_{D,point-like} = 1 + \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\frac{m_p^2}{\nu^2}\right)$$

$$\Delta E = -5.19745 * 0.8768^2 \simeq -4\ \mathrm{meV}$$

Definition of the proton radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0},$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

$$c_{D,point-like} = 1 + \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\frac{m_p^2}{\nu^2}\right)$$

$$\Delta E = -5.19745 * 0.8768^2 \simeq -4\ \mathrm{meV}$$

Definition of the neutron radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)\,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2}F_i' + \ldots$$

Definition of the neutron radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)$$

$$F_i(q^2) = 0 + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_n^2 = 6\frac{d}{dq^2}G_{n,E}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}c_D^{(n)}$$

$$c_D = 0 + 2F_2 + 8F_1' = 0 + 8m_n^2\left.\frac{dG_{n,E}(q^2)}{d\,q^2}\right|_{q^2=0}$$

Standard definition (corresponds to the experimental number):

$$r_n^2 = \frac{3}{4}\frac{1}{m_n^2}c_D$$

Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n}\left(\alpha c_D - \frac{2}{\pi}c_{3,NR}^{nl}\right) \sim D_d^{(n)had}$$

It is not proportional to the radius

Definition of the neutron radius

$$\langle p', s|J^\mu|p, s\rangle = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m_p}\right]u(p)$$

$$F_i(q^2) = 0 + \frac{q^2}{m_p^2}F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_n^2 = 6\frac{d}{dq^2}G_{n,E}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}c_D^{(n)}$$

$$c_D = 0 + 2F_2 + 8F_1' = 0 + 8m_n^2\left.\frac{dG_{n,E}(q^2)}{d\,q^2}\right|_{q^2=0}$$

Standard definition (corresponds to the experimental number):

$$r_n^2 = \frac{3}{4}\frac{1}{m_n^2}c_D$$

Neutron-lepton scattering length = REAL low energy constant

$$b_{nl} = \frac{1}{4m_n}\left(\alpha c_D - \frac{2}{\pi}c_{3,NR}^{nl}\right) \sim D_d^{(n)had}$$

It is not proportional to the radius

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t) \Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$- V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t) \Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x} \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma}\mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \ldots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma}\mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\right\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hadronic corrections: Spin-dependent

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} dt S^\dagger(\mathbf{x},\mathbf{X},t)\Big\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r}$$

$$-V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) + e\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Big\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,,$$

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.}(\mathbf{S}_1+\mathbf{S}_2)^2\delta^3(\mathbf{r})$$

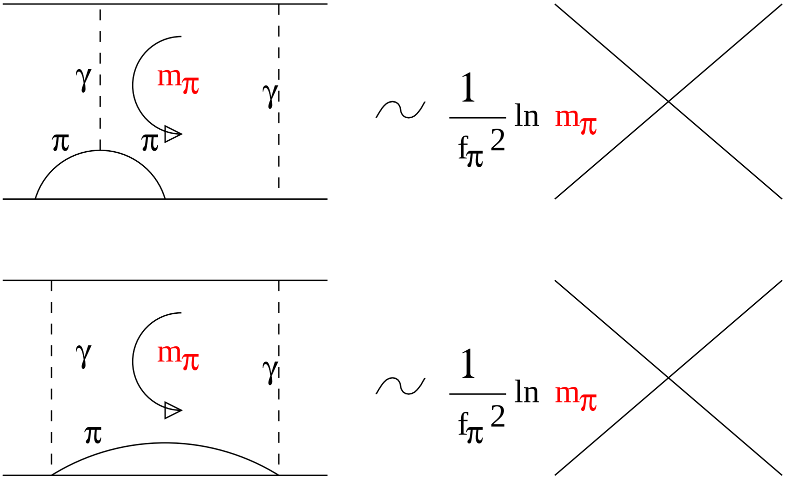
$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Leading chiral logs to the hyperfine splitting



$$\delta V = 2\frac{c_{4,NR}}{m_p^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r})\,.$$

c_4, Spin-dependent effects (Zemach): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

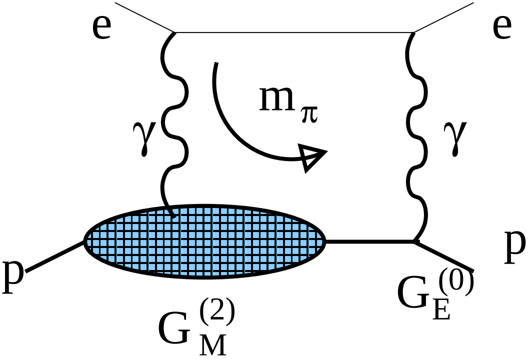


Figure : *Symbolic representation (plus permutations) of the Zemach correction.*

$$\delta c^{pl}_{4,Zemach} = (4\pi\alpha)^2 m_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(2)} .$$

c_4, Spin-dependent effects (Zemach): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

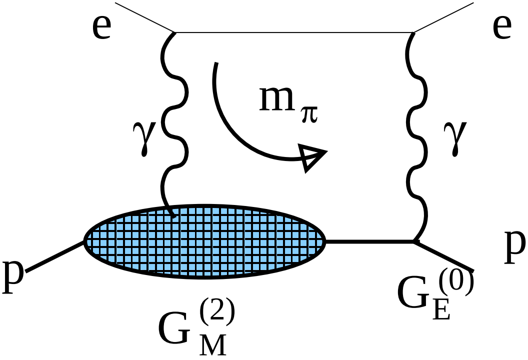


Figure : *Symbolic representation (plus permutations) of the Zemach correction.*

$$\delta c^{pl}_{4,Zemach} = (4\pi\alpha)^2 m_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(2)} .$$

c_4, Spin-dependent effects (polarizability): $\mathcal{O}(m_\mu\alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

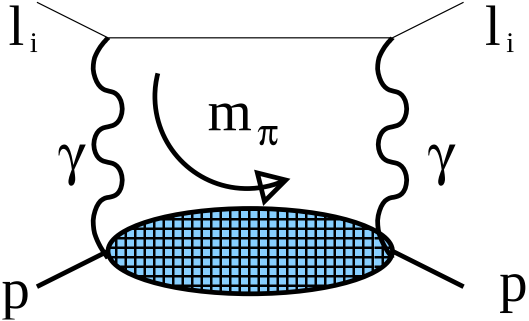


Figure : *Symbolic representation (plus permutations) of the spin-dependent polarizability correction.*

$$\delta c^{pl}_{4,pol} = -\frac{ig^4}{3}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{k^4-4m_l^2k_0^2}\left\{A_1(k_0,k^2)(k_0^2+2k^2)+3k^2\frac{k_0}{m_p}A_2(k_0,k^2)\right\}$$

c_4, Spin-dependent effects (polarizability): $\mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$

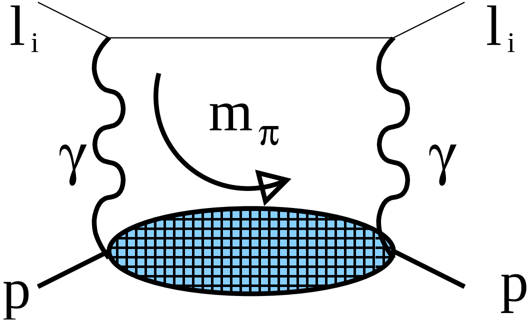


Figure : *Symbolic representation (plus permutations) of the spin-dependent polarizability correction.*

$$\delta c_{4,pol}^{pl} = -\frac{ig^4}{3} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{k^4 - 4m_l^2 k_0^2} \left\{ A_1(k_0, k^2)(k_0^2 + 2k^2) + 3k^2 \frac{k_0}{m_p} A_2(k_0, k^2) \right\}$$

$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned}
T^{\mu\nu} \quad = \quad & \left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2) \\
& +\frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2) \\
& -\frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}s_{\sigma}A_1(\rho,q^2) \\
& -\frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma}q_{\rho}\left((m_p\rho)s_{\sigma}-(q\cdot s)p_{\sigma}\right)A_2(\rho,q^2)
\end{aligned}$$

$$\delta c^{pl_i}_{4,point-like} = \frac{3+2c_F-c_F^2}{4}\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}\,.$$

$$\delta c^{pl_i}_{4,Zemach-u,d} \simeq \frac{m_p^2}{(4\pi F_0)^2}\alpha^2\frac{2}{3}\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2}\,,$$

$$\delta c^{pl_i}_{4,Zemach-\Delta} \simeq \frac{m_p^2}{(4\pi F_0)^2}\alpha^2\frac{8}{27}\pi^2 g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2}\,.$$

$$\delta c^{pl_i}_{4,pol.-\Delta} = \frac{b_{1,F}^2}{18}\alpha^2 \ln\frac{\Delta^2}{\nu^2}\,,$$

$$\delta c^{pl_i}_{4,pol.-\pi N} = -\frac{m_p^2}{(4\pi F_0)^2}g_A^2\frac{\alpha^2}{\pi}\frac{8}{3}C \ln\frac{m_\pi^2}{\nu^2}\,,$$

$$\delta c^{pl_i}_{4,pol.-\pi\Delta} = \frac{m_p^2}{(4\pi F_0)^2}g_{\pi N\Delta}^2\frac{\alpha^2}{\pi}\frac{64}{27}C \ln\frac{\Delta^2}{\nu^2}\,.$$

Only logarithmically chiral enhanced but they can be determined from hydrogen hyperfine splitting.

$$\begin{aligned}
\delta c_{4,NR}^{pl} \simeq & \left(1 - \frac{\mu_p^2}{4}\right)\alpha^2 \ln\frac{m_l^2}{\nu^2} \\
& + \frac{b_{1,F}^2}{18}\alpha^2 \ln\frac{\Delta^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{3}\left(\frac{2}{3}+\frac{7}{2\pi^2}\right)\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2} \\
& + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{8}{27}\left(\frac{5}{3}-\frac{7}{\pi^2}\right)\pi^2 g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2},
\end{aligned}$$

$$E_{\mathrm{HF}} = 4\frac{c_{4,NR}^{pl_i}}{m_p^2}\frac{1}{\pi}(\mu_{l_i p}\alpha)^3 \sim m_{l_i}\alpha^5\frac{m_{l_i}^2}{m_p^2} \times (\ln m_q, \ln\Delta, \ln m_{l_i}).$$

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\rm HF,logarithms}(m_\rho) = -0.031\ {\rm MHz}\,,$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\rm HF}(QED) - E_{\rm HF}(exp) = -0.046\ {\rm MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c^{pl}_{4,NR} = -47.7\alpha^2$ and $c^{pl}_{4,R}(m_\rho) \simeq c^{p}_{4,R}(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$\Delta E_{HF} \simeq -0.153 meV\ (Pachucki : -0.145)$$

$$\Delta E = \frac{1}{4}(-0.15) meV = -0.0375 meV$$

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\rm HF, logarithms}(m_\rho) = -0.031 \ {\rm MHz}\,,$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\rm HF}(QED) - E_{\rm HF}(exp) = -0.046 \ {\rm MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c^{pl}_{4,NR} = -47.7\alpha^2$ and $c^{pl}_{4,R}(m_\rho) \simeq c^{p}_{4,R}(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$\Delta E_{HF} \simeq -0.153 meV \ (Pachucki : -0.145)$$

$$\Delta E = \frac{1}{4}(-0.15) meV = -0.0375 meV$$

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\rm HF,logarithms}(m_\rho) = -0.031 \text{ MHz},$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\rm HF}(QED) - E_{\rm HF}(exp) = -0.046 \text{ MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c^{pl}_{4,NR} = -47.7\alpha^2$ and $c^{pl}_{4,R}(m_\rho) \simeq c^{p}_{4,R}(m_\rho) \simeq -16\alpha^2$.

Muonic hydrogen.

$$\Delta E_{HF} \simeq -0.153 meV \ (Pachucki : -0.145)$$

$$\Delta E = \frac{1}{4}(-0.15) meV = -0.0375 meV$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257 \frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257\frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257 \frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).

Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257 \frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257\,\frac{r_p^2}{\mathrm{fm}^2}\ \mathrm{meV},$$

$$r_p = 0.8424(22)\ \mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257\frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257\frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

CONCLUSIONS

Important to have a model independent and efficient approach to the problem. Effective Field Theories suitable for this task.

The proton radius is a matching coefficient of the effective theory. In general it is an scheme/scale dependent object.

Precise determination of hadronic parameters from alternative sources (experiment).
Non-trivial double checks by chiral perturbation theory.

Previous claims about r^3 unfounded.

Polarizabilities effects can only be computed in a model independent way with chiral perturbation theory. They can significantly reduce the error.

$$E_L^{ours} = 206.0791(190) - 5.2257 \frac{r_p^2}{\mathrm{fm}^2}\,\mathrm{meV},$$

$$r_p = 0.8424(22)\,\mathrm{fm}.$$

Beyond $m\alpha^5$. Resummation of logarithms

Pineda

$$\delta E^{\rm pot}_{n,l,j}(\nu_{us}) = E_n\alpha^2\frac{Z^2\delta_{l0}}{3n}\left(-\frac{16}{\beta_0}\log\left(\frac{\alpha(\nu_{us})}{\alpha}\right)-3(c_D-1)\right),$$

where $c_D(\nu_s)=1+\frac{16}{3}\ln z$, $z=\left[\frac{\alpha(\nu_s)}{\alpha(m)}\right]^{\frac{1}{\beta_0}}\simeq 1-1/(2\pi)\alpha(\nu_s)\ln(\frac{\nu_s}{m})$, $\beta_0=-\frac{4}{3}T_F n_f$ with $T_F=1$, $E_n=-mZ^2\alpha^2/(2n^2)$, $\nu_s=2a_n^{-1}$, where $a_n^{-1}=\frac{mZ\alpha(2a_n^{-1})}{n}$. α is also understood at the soft scale $\nu_s=2a_n^{-1}$ unless the scale is specified. We take $\nu_{us}=-E_n$.

$$\begin{aligned} D_d^{(2)}-\frac{\alpha(\nu)}{2} &= \frac{\alpha(\nu_{us})}{2}\left(1+\frac{\beta_0}{2\pi}\alpha(\nu_{us})\ln\frac{mv^2}{mv}+\cdots\right)\\ &\times\left(-\frac{8}{3}\frac{\alpha(\nu_{us})}{\pi}\ln\frac{mv^2}{m}-\frac{2}{3}\beta_0\left(\frac{\alpha(\nu_{us})}{\pi}\right)^2\ln^2\frac{mv^2}{m}+\cdots\right)\\ &\simeq -\frac{4}{3}\frac{\alpha^2(\nu_{us})}{\pi}\ln\frac{mv^2}{m}-\frac{2\beta_0}{3}\frac{\alpha^3(\nu_{us})}{\pi^2}\ln\frac{mv^2}{mv}\ln\frac{mv^2}{m}\\ &\quad -\frac{1}{3}\beta_0\frac{\alpha^3(\nu_{us})}{\pi^2}\ln^2\frac{mv^2}{m}. \end{aligned}$$

It agrees with Pachucki.