Kinematic power corrections in off-forward hard processes

A. N. Manashov

University of Regensburg

based on

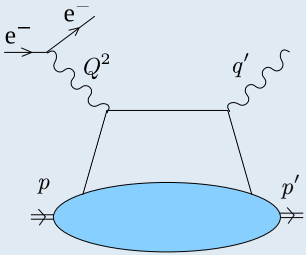
V. Braun, A. Manashov: PRL 107 (2011) 202001;
JHEP 1201 (2012) 085

V. Braun, A. Manashov, B. Pirnay: PRD86 (2012) 014003;
arXiv:1209.2559

Bochum, 18.04.2013

Hard exclusive processes involve off-forward matrix elements

DVCS: $\gamma^* N(p) \to \gamma N(p')$ **D. Müller, X. Ji, A. Radyushkin**



Operator Product Expansion

$$\mathcal{A}_{\mu\nu} \sim \langle p'|j_\mu^{em}(x) j_\nu^{em}(0)|p\rangle \quad \sim \quad \sum_N C_{\mu\nu}^N(x^2,\mu^2)\,\langle p'|\mathcal{O}_N(\mu^2)|p\rangle$$

Kinematic variables: hadron mass m^2 momentum transfer $t = (P - P')^2$

How to calculate effects $\sim m^2/Q^2$ and t/Q^2?

current conservation

twist-3: Anikin, Teryaev; Belitsky, Müller; Kivel, Polyakov, Schäfer, Teryaev, 2001

$$T\{j_\mu(z_1 x) j_\nu(z_2 x)\} = T^{t=2}_{\mu\nu}(z_1, z_2) + T^{t=3}_{\mu\nu}(z_1, z_2) + T^{t=4}_{\mu\nu}(z_1, z_2) + \dots$$

$$\mathcal{A}_{\mu\nu}(p, p', q) = i \int d^4x \, e^{-i(z_1 q - z_2 q')} \langle p' | T\{j_\mu(z_1 x) j_\nu(z_2 x)\} | p \rangle$$

$z_1 - z_2 = 1.$

Conservation of the electromagnetic current:

$$q^\mu \mathcal{A}_{\mu\nu}(p, p', q) = (q')^\nu \mathcal{A}_{\mu\nu}(p, p', q) = 0$$

only valid in the sum of all twists but not for each twist separately

Kinematic power corrections

OPE = expansion over multiplicatively renormalized operators

$$j_\mu^{em}(x) = \bar{q}(x)\,Q\gamma_\mu\,q(x)$$

$$T\{j_\mu^{em}(x)j_\nu^{em}(0)\} \quad = \quad \sum_N C_{\mu\nu}^N(x)\,\mathcal{O}_N(0) \; + \; \text{higher twists}$$

$\mathcal{O}_N$ − twist two operators: $\quad (\mathcal{O}_N^{\mu_1\cdots\mu_{N+1}} = \bar{q}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_{N+1}}\,q)$

Higher twist operators related to $\mathcal{O}_N$:

- twist−3: $\quad \left[\mathbf{P}_{[\mu}\,\mathcal{O}_{\mu_1]\ldots\mu_{N+1}}\right]$
- twist−4: $\quad \left[\mathbf{P}_\mu\left[\mathbf{P}^\mu\,\mathcal{O}_{\mu_1\ldots\mu_{N+1}}\right]\right], \qquad \left[\mathbf{P}^\mu, \mathcal{O}_{\mu\ldots\mu_{N+1}}\right]$

"Kinematical power corrections" = taking into account the contributions of these operators to the OPE only

General structure

$$T\{j(x)j(0)\} = \sum_N a_N \mathcal{O}_N + \sum_N \Big(b_N \partial^2 \mathcal{O}_N + c_N (\partial\mathcal{O})_N \Big) + \text{all others}$$

all others $\neq$ all quark-gluon operators

To determine the coefficients a_N, b_N in the leading order in α_s it is sufficient to take the matrix elements over free quarks. But it does not work for the coefficients c_N.

S. Ferrara, A. F. Grillo, G. Parisi and R. Gatto, Phys. Lett. **B38**, 333 (1972):

matrix elements of $\partial^\mu \mathcal{O}_{\mu\mu_1\ldots\mu_N}$ over free quarks vanish

The same problem in a different language

- **Using EOM $\partial^\mu \mathcal{O}_{\mu\mu_1\ldots\mu_N}$ can be expressed in terms of quark-gluon operators:**

$N = 1$:

$$\partial^\mu O_{\mu\nu} = 2i\bar{q}gF_{\nu\mu}\gamma^\mu q\,,$$

where

$$O_{\mu\nu} = (1/2)[\bar{q}\gamma_\mu \overleftrightarrow{D}_\nu q + (\mu \leftrightarrow \nu)]$$

$N = 2$:

$$\begin{aligned}\frac{4}{5}\partial^\mu \mathcal{O}_{\mu\alpha\beta} =& -12i\bar{q}\gamma^\rho \left\{ gF_{\rho\beta}\overrightarrow{D}_\alpha - \overleftarrow{D}_\alpha gF_{\rho\beta} + (\alpha \leftrightarrow \beta)\right\} q \\ & - 4\partial^\rho \bar{q}(\gamma_\beta g\widetilde{F}_{\alpha\rho} + \gamma_\alpha g\widetilde{F}_{\beta\rho})\gamma_5 q \\ & - \frac{8}{3}\partial_\beta \bar{q}\gamma^\sigma \widetilde{F}_{\sigma\alpha}\gamma_5 q - \frac{8}{3}\partial_\alpha \bar{q}\gamma^\sigma \widetilde{F}_{\sigma\beta}\gamma_5 q + \frac{28}{3} g_{\alpha\beta}\partial_\rho \bar{q}\gamma^\sigma \widetilde{F}_{\sigma\rho} q\,,\end{aligned}$$

where

$$\mathcal{O}_{\mu\alpha\beta} = \mathrm{Sym}_{\mu\alpha\beta}\left[\frac{15}{2}\bar{q}\gamma_\mu \overleftrightarrow{D}_\alpha \overleftrightarrow{D}_\beta\, q - \frac{3}{2}\partial_\alpha\partial_\beta \bar{q}\gamma_\mu q\right] - \mathrm{traces}$$

etc.

General structure

Let G_{Nk} be a complete basis of twist-four operators

$$(D^n\bar{q})(D^m F)(D^l q), \qquad (D^n\bar{q})(D^m F)(D^p F)(D^l q), \ldots$$

EOM imply that some linear combination of quark gluon operators G_{Nk} is related to leading twist:

$$(\partial\mathcal{O})_N = \sum_k a_k^{(N)}\, G_{Nk}$$

$\hookrightarrow$ exact relation between matrix elements

Twist-4 contribution to OPE

$$T\{j(x)j(0)\}^{\text{tw}-4} = \sum_{N,k} c_{N,k}(x)\, G_{N,k} + \ldots \stackrel{?}{=} \sum_N \tilde{c}_N (\partial\mathcal{O})_N + \ldots$$

Local & nonlocal operators

Conformal operator

$$\mathcal{O}_N(x) = (\partial_{z_1} + \partial_{z_2})^N C_N^{(\frac{3}{2})}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right) \bar{q}(x + z_1 n)\gamma_+ q(x + z_2 n)\Big|_{z_1=z_2=0}$$

Too complicated!

Nonlocal operator

$$\begin{aligned} O_{++}(x; z_1, z_2) &= \bar{q}(x + z_1 n)\gamma_+ q(x + z_2 n) \\ &= \sum_{nm} z_1^n z_2^m O_{nm}(x) = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\mathcal{O}_{Nk}(x)\,. \end{aligned}$$

$$O_{nm} = \frac{1}{n!m!}\partial_+^n \bar{q}(x)\partial_+^m q(x)\,, \qquad \mathcal{O}_{Nk}(x) = \sum_{nm} A_{nm}^{Nk} O_{nm}.$$

$\mathcal{O}_{Nk}(x)$–"basis vectors", $\Psi_{Nk}(z_1, z_2)$– "coordinates".

Conformal basis: $\mathcal{O}_N(x)$ - conformal operator $\delta_K \mathcal{O}_N(x) = 0$

$$\mathcal{O}_{Nk}(x) = \partial_+^k \mathcal{O}_N(x)$$

At one loop : conformal operator = multiplicatively renormalized operator

Local & nonlocal operators

conformal expansion

$$O_{++}(x; z_1, z_2) = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\mathcal{O}_{Nk}(x)\,.$$

$$[i(\mathbf{D} + \mathbf{M}_{+-}), O_{Nk}] = (N+k)\,O_{Nk}\,, \qquad [i(n\mathbf{P}), O_{Nk}] = O_{Nk+1}\,, \qquad [i(\bar{n}\mathbf{K}), O_{Nk}] \sim O_{Nk-1}$$

$$[i(\mathbf{D} + \mathbf{M}_{+-}), O_{++}(x; z_1, z_2)] = S_0\, O_{++}(x; z_1, z_2)\,,$$

$$[i(n\mathbf{P}), O_{++}(x; z_1, z_2)] = -S_-\, O_{++}(x; z_1, z_2)\,,$$

$$[i(\bar{n}\mathbf{K}), O_{++}(x; z_1, z_2)] = (n\bar{n})S_+\, O_{++}(x; z_1, z_2)$$

$$S_- = -\partial_{z_1} - \partial_{z_2}\,, \qquad S_0 = z_1\partial_{z_1} + z_2\partial_{z_2} + 2\,, \qquad S_+ = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2)\,.$$

S_+, S_0, S_- **are the generators of** $sl(2)$ **algebra**

$$[S_+, S_-] = 2S_0\,, \qquad [S_0, S_\pm] = \pm S_\pm$$

Local & nonlocal operators

- $\Psi_{Nk}(z_1, z_2)$ are polynomials of degree $N+k$ in z_1, z_2
- $\Psi_{N0}(z_1, z_2) \sim (z_1 - z_2)^N$ $\quad\quad \Big[(\partial_{z_1} + \partial_{z_2})\Psi_{Nk}(z_1, z_2) = \Psi_{Nk-1}(z_1, z_2)\Big]$
- $S_+\Psi_{Nk}(z_1, z_2) \sim \Psi_{Nk+1}(z_1, z_2)$ $\quad\quad \Big[\Psi_{Nk} \sim S_+^k (z_1 - z_2)^N\Big]$

$$(n\partial_x) O_{++}(x; z_1, z_2) = (\partial_{z_1} + \partial_{z_2}) O_{++}(x; z_1, z_2)\,.$$

$$\sum_{Nk} \Psi_{Nk}(z_1, z_2)\partial_+^{k+1}\mathcal{O}_N(x) = \sum_{Nk} (\partial_{z_1} + \partial_{z_2})\Psi_{Nk}(z_1, z_2)\partial_+^{k}\mathcal{O}_N(x)\,.$$

$$\delta_K \partial_+^k \mathcal{O}_N \sim \partial_+^{k-1}\mathcal{O}_N \Longrightarrow S_+\Psi_{Nk}(z_1, z_2) \sim \Psi_{Nk+1}(z_1, z_2)\,.$$

RG-equation

$$\mu\frac{d}{d\mu} O_{++}(x; z_1, z_2) = [\mathbb{H} O_+](x; z_1, z_2) \Longrightarrow [\mathbb{H}\Psi_{Nk}](z_1, z_2) = \gamma_N \Psi_{Nk}(z_1, z_2)$$

How to project out $\mathcal{O}_N(x)$ from nonlocal operator?

Conformal symmetry and $SU(1,1)$ scalar product

collinear conformal transformations $\quad SL(2,\mathbb{R}) \Leftrightarrow SU(1,1)$

$$x_\mu = z n_\mu\,, \qquad z\in\mathbb{R}\to z' = \frac{az+b}{cz+d}\,, \qquad \Leftrightarrow \qquad z\in\mathbb{C}\to z' = \frac{az+b}{\bar b z+\bar a}$$

representations are labeled by conformal spin

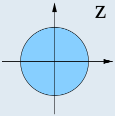
$$\varphi(z)\to T^j\varphi(z) = \frac{1}{(\bar b z+\bar a)^{2j}}\varphi\left(\frac{az+b}{\bar b z+\bar a}\right)$$

This is a unitary transformation with respect to the following scalar product:

$$\langle\phi,\psi\rangle_j = \frac{2j-1}{\pi}\int_{|z|<1} d^2z\,(1-|z|^2)^{2j-2}\bar\phi(z)\psi(z) \equiv \int_{|z|<1}\mathcal{D}_j z\,\bar\phi(z)\psi(z)\,, \qquad ||\phi||^2 = \langle\phi,\phi\rangle$$

similar for several variables

$$\langle\phi,\psi\rangle_{j_1,j_2} = \int_{|z_1|<1}\mathcal{D}_{j_1}z_1\int_{|z_2|<1}\mathcal{D}_{j_2}z_2\,\bar\phi(z_1,z_2)\psi(z_1,z_2)$$



Generators

- Generators of infinitesimal $SU(1,1)$ transformations

$$S_+ = z^2\partial_z + 2jz\,, \qquad S_0 = z\partial_z + j\,, \qquad S_- = -\partial_z$$

satisfy the usual $SL(2)$ algebra

$$[S_+, S_-] = 2S_0\,, \qquad [S_0, S_\pm] = \pm S_\pm$$

- For products of fields, e.g. $\phi(z_1)\phi(z_2)\ldots$

$$S_+^{(j_1,j_2)} = S_+^{(j_1)} + S_+^{(j_2)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2j_1 z_1 + +2j_2 z_2\,,$$

$$S_+^{(j_1,j_2,j_3)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + z_3^2\partial_{z_3} + 2j_1 z_1 + +2j_2 z_2 + 2j_3 z_3\,,$$

- Hermiticity properties

$$S_0^\dagger = S_0 \qquad (S_+)^\dagger = -S_- \qquad \langle\phi, S\psi\rangle = \langle S^\dagger\phi, \psi\rangle$$

Some properties

- different powers are orthogonal

$$\langle z^n, z^{n'}\rangle = \delta_{nn'}\,||z^n||^2\,,$$

- reproducing kernel (unit operator)

$$\phi(z) \;=\; \int_{|w|<1} \mathcal{D}_j w\, \mathcal{K}_j(z,\bar{w})\phi(w)\,, \qquad \mathcal{K}_j(z,\bar{w}) = \frac{1}{(1-z\bar{w})^{2j}}$$

- Fourier

$$\rho_N \iint_{|z_i|<1} \mathcal{D}_1 z_1 \mathcal{D}_1 z_2\, (\bar{z}_1-\bar{z}_2)^N e^{ip_1 z_1 + ip_2 z_2} \;=\; i^N (p_1+p_2)^N C_N^{3/2}\left(\frac{p_1-p_2}{p_1+p_2}\right)$$

Conformal operators vs. Light-ray operators

- Light-ray operators

$$O_{++}(z_1, z_2) = \bar{q}(z_1 n)\,\gamma_+\, q(z_2 n)$$

- (Local) conformal operators

$$\mathcal{O}_N(x) \quad = \quad (-\partial_+)^N \bar{q}(x)\gamma_+ C_N^{3/2}\left(\frac{\overrightarrow{D}_+ - \overleftarrow{D}_+}{\overrightarrow{D}_+ + \overleftarrow{D}_+}\right) q(x)$$

$\rightarrow$ Projecting $\mathcal{O}_N$ from $O_+(z_1, z_2)$:

$$\mathcal{O}_N(x) \quad = \quad (\partial_{z_1} + \partial_{z_2})^N C_N^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right) O_{++}(z_1 n + x, z_2 n + x)\Big|_{z_i=0}$$

- Alternatively:

$$\rho_N = \frac{1}{2}(N+1)(N+2)!$$

$$\mathcal{O}_N = \rho_N \left\langle z_{12}^N, O_{++}(z_1, z_2)\right\rangle_{11} = \rho_N \iint_{|z_i|<1} \mathcal{D}_1 z_1 \mathcal{D}_1 z_2\, \bar{z}_{12}^N\, O_{++}(z_1, z_2)$$

Conformal operators vs. Light-ray operators

$$\delta_K O_{++}(z_1, z_2) = 2(n\bar{n})\, S_+^{(1,1)} O_{++}(z_1, z_2)$$

$$\begin{aligned}
\delta_K \mathcal{O}_N &= \rho_N \iint_{|z_i|<1} \mathcal{D}z_1 \mathcal{D}z_2\, \bar{z}_{12}^N\, \delta_K\, O_{++}(z_1, z_2) \sim \iint_{|z_i|<1} \mathcal{D}z_1 \mathcal{D}z_2\, \bar{z}_{12}^N\, S_+^{(1,1)}\, O_{++}(z_1, z_2) \\
&= -\iint_{|z_i|<1} \mathcal{D}z_1 \mathcal{D}z_2 \left(S_-^{(1,1)}\, z_{12}^N \right)^* O_{++}(z_1, z_2) = 0\,,
\end{aligned}$$

Divergency $(\partial\mathcal{O})_N$

$$\begin{aligned}
(\partial\mathcal{O})_N^{free} &= \rho_N \iint_{|z_i|<1} \mathcal{D}z_1 \mathcal{D}z_2\, \bar{z}_{12}^N\, \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial n_\mu} O_{++}(x; z_1, z_2) \\
&\sim \iint_{|z_i|<1} \mathcal{D}z_1 \mathcal{D}z_2\, \bar{z}_{12}^N\, S_+^{(1,1)}\, \partial_\mu \bar{q}(x + z_1 n) \gamma_+ \partial^\mu q(x + z_2 n) = 0
\end{aligned}$$

- Symmetry generators, S_α
- Invariant scalar product
- Hermiticity of evolution kernel $\mathbb{H} = \mathbb{H}^\dagger$

$$O_{++}(x; z_1, z_2) = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\mathcal{O}_{Nk}(x)\,.$$

$$\mathcal{O}_{Nk}(x) = \frac{1}{||\Psi_{Nk}||^2}\langle \Psi_{Nk}(z_1, z_2)|O_{++}(x; z_1, z_2)\rangle\,.$$

Spinor Representation

Coordinates:

$$x_{\alpha\dot\alpha} = x_\mu \sigma^\mu_{\alpha\dot\alpha} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & w \\ \bar w & x_- \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}, \vec\sigma)$$

To maintain Lorentz–covariance, introduce two light-like vectors $n^2 = \tilde n^2 = 0$

$$n_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}, \qquad \tilde n_{\alpha\dot\alpha} = \mu_\alpha \bar\mu_{\dot\alpha}$$

with auxiliary spinors λ and μ

$$x_{\alpha\dot\alpha} = z\,\lambda_\alpha\bar\lambda_{\dot\alpha} + \tilde z\,\mu_\alpha\bar\mu_{\dot\alpha} + w\,\lambda_\alpha\bar\mu_{\dot\alpha} + \bar w\,\mu_\alpha\bar\lambda_{\dot\alpha}$$

Fields:

$$q = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}, \qquad \bar q = (\chi^\beta, \bar\psi_{\dot\alpha}),$$

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}\right)$$

$f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ transform according to $(1,0)$ and $(0,1)$ representations of Lorentz group

“Plus” and “Minus” projections

$$\psi_+ = \lambda^\alpha \psi_\alpha\,, \qquad \chi_+ = \lambda^\alpha \chi_\alpha\,, \qquad f_{++} = \lambda^\alpha \lambda^\beta f_{\alpha\beta}\,,$$
$$\bar\psi_+ = \bar\lambda^{\dot\alpha} \bar\psi_{\dot\alpha}\,, \qquad \bar\chi_+ = \bar\lambda^{\dot\alpha} \chi_{\dot\alpha}\,, \qquad \bar f_{++} = \bar\lambda^{\dot\alpha} \bar\lambda^{\dot\beta} \bar f_{\dot\alpha\dot\beta}\,,$$
$$\psi_- = \mu^\alpha \psi_\alpha\,, \qquad \bar\psi_- = \bar\mu^{\dot\alpha} \bar\psi_{\dot\alpha} \qquad f_{+-} = \lambda^\alpha \mu^\beta f_{\alpha\beta}$$

Conformal basis for twist-four non-quasipartonic operators

Braun, Manashov, Rohrwild, Nucl. Phys. **B826** (2010) 235.

$$Q_1(z_1,z_2,z_3) = \bar\psi_+(z_1) f_{+-}(z_2)\psi_+(z_3)\,, \qquad T^{j=1}\otimes T^{j=1}\otimes T^{j=1}$$

$$Q_2(z_1,z_2,z_3) = \bar\psi_+(z_1) f_{++}(z_2)\psi_-(z_3)\,, \qquad T^{j=1}\otimes T^{j=3/2}\otimes T^{j=1/2}$$

$$Q_3(z_1,z_2,z_3) = \frac{1}{2}[D_{-+}\bar\psi_+](z_1) f_{++}(z_2)\psi_+(z_3)\,, \qquad T^{j=3/2}\otimes T^{j=3/2}\otimes T^{j=1}$$

and three similar operators with $f \to \bar f$

cf. in usual notation

$$\bar q_L(z_1)\Big[F_{+\mu}(z_2) + i\widetilde F_{+\mu}(z_2)\Big]\gamma^\mu q_L(z_3) = Q_2(z_1,z_2,z_3) - Q_1(z_1,z_2,z_3)$$

$$\bar q_L(z_1)\Big[F_{+\mu}(z_2) - i\widetilde F_{+\mu}(z_2)\Big]\gamma^\mu q_L(z_3) = \bar Q_2(z_1,z_2,z_3) - \bar Q_1(z_1,z_2,z_3)$$

$$\vec Q(z_1,z_2,z_3) = \begin{pmatrix} Q_1(z_1,z_2,z_3) \\ Q_2(z_1,z_2,z_3) \\ Q_3(z_1,z_2,z_3)\end{pmatrix}$$

Divergence of a conformal operator

The conformal operator $\mathcal{O}_N$ is obtained by the projection

$$\mathcal{O}_N = n^{\mu} n^{\mu_1} \dots n^{\mu_N} (\mathcal{O}_N)_{\mu\mu_1\dots\mu_N}$$

where n is an auxiliary light-like vector. Define a divergence of the conformal operator as

$$(\partial\mathcal{O})_N \quad = \quad n^{\mu_1} \dots n^{\mu_N} \partial^{\mu} (\mathcal{O}_N)_{\mu\mu_1\dots\mu_N} = n^{\mu_1} \dots n^{\mu_N} \Big[i\mathbf{P}^{\mu}, (\mathcal{O}_N)_{\mu\mu_1\mu_2\dots\mu_N} \Big]$$

where $\mathbf{P}_\mu$ is the usual four-momentum operator

$$\mathbf{P}_\mu |p\rangle = p_\mu |p\rangle \,, \qquad i[\mathbf{P}_\mu, \Phi(x)] = \frac{\partial}{\partial x^\mu} \Phi(x)$$

Taking into account that $n^\mu = \frac{1}{2}(\lambda\sigma^\mu\bar{\lambda})$ the same definition can be rewritten as

$$(\partial\mathcal{O})_N = \frac{1}{N+1} \left[i\mathbf{P}^{\mu}, \frac{\partial}{\partial n^\mu} \mathcal{O}_N(n) \right] = \frac{1}{(N+1)^2} \left[i\bar{\mathbf{P}}^{\dot\alpha\alpha}, \frac{\partial}{\partial\lambda^\alpha} \frac{\partial}{\partial\bar{\lambda}^{\dot\alpha}} \mathcal{O}_N(\lambda,\bar{\lambda}) \right],$$

Divergence of a conformal operator

$$O_{++}(z_1, z_2) = \bar{\psi}_+(z_1 n)[z_1 n, z_2 n]\psi_+(z_2 n)$$

Let

$$(\partial O)_{++}(z_1, z_2) = \left[i\bar{\mathbf{P}}^{\dot\alpha\alpha}, \frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\bar\lambda^{\dot\alpha}}\mathcal{O}_{++}(z_1, z_2)\right]$$

then

$$(\partial\mathcal{O})_N = \frac{\rho_N}{(N+1)^2}\left\langle z_{12}^N, (\partial O)_{++}(z_1, z_2)\right\rangle \equiv \frac{\rho_N}{(N+1)^2}\iint_{|z_i|<1}\mathcal{D}z_1\mathcal{D}z_2\,\bar{z}_{12}^N\,(\partial O)_{++}(z_1, z_2)$$

By a direct calculation obtain

$$2(\partial\mathcal{O})_{++}(z_1, z_2) = ig\Big[A(z_1, z_2) - \bar{A}(z_1, z_2)\Big] + \dots\,,$$

$$\begin{aligned} A(z_1, z_2) \quad = \quad & \partial_{z_2} z_{12}^2\Big\{Q_1(z_1, z_1, z_2) + \int_0^1 du\, u\Big[\,Q_2(z_1, z_{21}^u, z_2) + z_{12}\,Q_3(z_1, z_{21}^u, z_2)\Big]\Big\} \\ & + \partial_{z_1}\partial_{z_2} z_{12}^3\int_0^1 du\,\Big[-Q_1(z_1, z_{21}^u, z_2) + \bar{u}\,Q_2(z_1, z_{21}^u, z_2)\Big] \end{aligned}$$

The ellipses stand for EOM, contributions of quasipartonic operators and terms $\propto S_{12}^+$ which do not contribute to the projection that defines the conformal operator

T-product of two electromagnetic currents

$$T_{\alpha\beta\dot\alpha\dot\beta}(z_1,z_2) = -\frac{2}{\pi^2 x^4 z_{12}^3}\Big\{x_{\alpha\dot\beta}\mathfrak{B}_{\beta\dot\alpha}(z_1,z_2) - x_{\beta\dot\alpha}\mathfrak{B}_{\alpha\dot\beta}(z_2,z_1) + x_{\alpha\dot\beta}x_{\beta\dot\alpha}\,\mathbb{A}(z_1,z_2) + x^2\Big[x_{\beta\dot\beta}\partial_{\alpha\dot\alpha}\mathbb{C}(z_1,z_2) - x_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\mathbb{C}(z_2,z_1)\Big] + \ldots\Big\}$$

$$\begin{aligned}
\mathfrak{B}_{\alpha\dot\alpha}(z_1,z_2) &= \mathfrak{B}^{t=2}_{\alpha\dot\alpha}(z_1,z_2) + \mathfrak{B}^{t=3}_{\alpha\dot\alpha}(z_1,z_2) + \mathfrak{B}^{t=4}_{\alpha\dot\alpha}(z_1,z_2) + \ldots \\
\mathbb{A}(z_1,z_2) &= \mathbb{A}^{t=4}(z_1,z_2) \\
\mathbb{C}(z_1,z_2) &= \mathbb{C}^{t=4}(z_1,z_2)
\end{aligned}$$

all twist four functions can be expressed in terms of

$$R(z_1,z_2) = ig\int_{z_2}^{z_1} dw(w-z_2)\,Q_2(z_1,w,z_2)$$

For example

$$\mathfrak{B}^{t=2}_{\alpha\dot\alpha}(z_1,z_2) = \frac{1}{2}\partial_{\alpha\dot\alpha}\int_0^1 du\,\mathcal{O}_{++}(uz_1,uz_2)$$

$$\mathbb{C}^{t=4}(z_1,z_2) = -\frac{1}{8}\int_0^1 \frac{du}{u^2}\Big[R(uz_1,uz_2) + \bar R(uz_2,uz_1)\Big]$$

Renormalization group equations

Braun, Manashov, Rohrwild, Nucl. Phys. **B826** (2010) 235.

The complete RG equation for twist-four operators has the following structure:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}=-\frac{\alpha_s}{2\pi}\begin{pmatrix}H_{11}&H_{12}&H_{13}\\H_{21}&H_{22}&H_{23}\\H_{31}&H_{32}&H_{33}\end{pmatrix}\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}$$

$$\overrightarrow{Q}(z_1,z_2,z_3)=\begin{pmatrix}Q_1(z_1,z_2,z_3)\\Q_2(z_1,z_2,z_3)\\Q_3(z_1,z_2,z_3)\end{pmatrix}=\sum_{Np}\vec{\Psi}_{Np}(z_1,z_2,z_3)\mathcal{Q}_{Np}$$

1) Symmetry generators, S_α 2) Invariant scalar product 3) Hermiticity of evolution kernel $\mathbb{H}=\mathbb{H}^\dagger$

$$\vec{S}_\alpha=\begin{pmatrix}S_\alpha^{(111)}&0&0\\0&S_\alpha^{(1\frac{3}{2}\frac{1}{2})}&0\\0&0&S_\alpha^{(\frac{3}{2}\frac{3}{2}1)}\end{pmatrix}$$

Scalar product

$$\langle\langle\overrightarrow{\Phi},\overrightarrow{\Psi}\rangle\rangle=2\langle\Phi_1,\Psi_1\rangle_{111}+\langle\Phi_2,\Psi_2\rangle_{1\frac{3}{2}\frac{1}{2}}+\frac{1}{2}\langle\Phi_3,\Psi_3\rangle_{\frac{3}{2}\frac{3}{2}1},$$

$$\mathcal{Q}_{Np}=\langle\vec{\Psi}_{Np}|\overrightarrow{Q}\rangle$$

Coefficient functions

$$(\partial\mathcal{O})_N \quad = \quad \frac{ig\rho_N}{(N+1)^2}\Big[\langle z_{12}^N, A(z_1,z_2)\rangle_{11} - \langle z_{12}^N, \bar{A}(z_1,z_2)\rangle_{11}\Big]$$

We want to rewrite this answer as a sum of terms

$$(\partial\mathcal{O})_N \sim \langle \Psi_i^N(w_1,w_2,w_3),\, Q_i(w_1,w_2,w_3)\rangle_{j_1 j_2 j_3}\,,$$

where j_1, j_2, j_3 are the conformal spins of the Q_i, so that the functions $\Psi_i^N(w_1,w_2,w_3)$ can be identified with the coefficient functions of the Q-operators

Consider the first two terms as an example:

$$A = \partial_{z_2} z_{12}^2 \Big[\phi_1(z_1,z_2) + \phi_2(z_1,z_2) + \ldots\Big] + \ldots$$

where

$$\phi_1(z_1,z_2) = Q_1(z_1,z_1,z_2) \qquad\qquad \phi_2(z_1,z_2) = \int_0^1 du\, u\, Q_2(z_1, z_{21}^u, z_2)$$

- reproducing kernel – the operator which does nothing

$$\phi(z) = \int_{|w|<1} \mathcal{D}_j w\, \mathcal{K}_j(z,\bar{w})\phi(w)\,, \qquad \mathcal{K}_j(z,\bar{w}) = \frac{1}{(1-z\bar{w})^{2j}}$$

$$Q_i(z_1,z_2,z_3) = \Big\langle \prod_{k=1}^{3} \mathcal{K}_{j_k}(z_k,\bar{w}_k),\, Q_i(w_1,w_2,w_3)\Big\rangle_{j_1 j_2 j_3}\,,$$

$$(\partial\mathcal{O})_N \sim \langle z_{12}^N, A(z_1,z_2)\rangle_{11} \sim \langle z_{12}^N, \partial_2 z_{12}^2\, Q_1(z_1,z_1,z_2)\rangle_{11} + \ldots$$

$$= \langle \Psi_1(w_1,w_2,w_3),\, Q_1(w_1,w_2,w_3)\rangle_{111}$$

$$\Psi_1(w_1,w_2,w_3) = \langle z_{12}^N, \partial_2 z_{12}^2 \mathcal{K}_1(z_1,\bar{w}_1)\mathcal{K}_1(z_1,\bar{w}_2)\mathcal{K}_1(z_3,\bar{w}_3)\rangle_{11}$$

Coefficient functions

Final results:

$$\Psi_N^{(1)}(w) = 4a_N\left[\int_0^1 d\alpha\,\bar\alpha\int_0^1 d\beta\,\bar\beta\,(w_{12}^{\alpha}-w_{32}^{\beta})^{N-1}-\frac{1}{N+1}\int_0^1 d\alpha\,\alpha\bar\alpha\,(w_{12}^{\alpha}-w_3)^{N-1}\right]$$

$$\Psi_N^{(2)}(w) = -4a_N\int_0^1 d\alpha\,\bar\alpha\int_0^1 d\beta\,\left(\beta+\frac{1}{N+1}\alpha\right)(w_{12}^{\alpha}-w_{32}^{\beta})^{N-1}$$

$$\Psi_N^{(3)}(w) = -24c_N\int_0^1 d\alpha\,\bar\alpha^2\alpha\int_0^1 d\beta\,\bar\beta\,(w_{12}^{\alpha}-w_{32}^{\beta})^{N-2}$$

where

$$a_N = \frac{1}{8}(N+3)(N+2)(N+1)N,$$

$$b_N = -\frac{1}{6}(N+3)(N+2)N$$

$$c_N = \frac{1}{48}\frac{(N+4)!}{(N+1)(N-2)!}$$

Coefficient functions

Calculation of the norm is straightforward

$$||\overrightarrow{\Psi}_N||^2 = 2||\Psi_N^{(1)}||^2_{111} + ||\Psi_N^{(2)}||^2_{1\frac{3}{2}\frac{1}{2}} + \frac{1}{2}||\Psi_N^{(3)}||^2_{\frac{3}{2}\frac{3}{2}1}\,.$$

$$||\overrightarrow{\Psi}_N||^2 = \frac{1}{2}||z_{12}^N||^2_{11}\,(N+2)^2(N+1)^2\left[\psi(N+3)+\psi(N+1)-\psi(3)-\psi(1)\right]$$

Compare: leading twist anomalous dimension

$$\gamma_N = C_F\left(1-\frac{2}{(N+1)(N+2)}+4\sum_{m=2}^{N+1}\frac{1}{m}\right) = 2C_F\left[\psi(N+3)+\psi(N+1)-\psi(3)-\psi(1)\right]$$

$$ig\,\overrightarrow{Q}(z_1, z_2, z_3) \quad = \quad \sum_{N=0}^{\infty}\sum_{k=0}^{\infty} \frac{p_{Nk}(N+1)^2}{\rho_N ||\Psi_N||^2}\, \overrightarrow{\Psi}_{Nk}(z_1, z_2, z_3)\, \partial_+^k (\partial\mathcal{O})_N + \dots$$

- All entries known explicitly
- The ellipses stand for “dynamic” operators

$$R(z_1, z_2) = ig \int_{z_2}^{z_1} dw(w-z_2)\, Q_2(z_1, w, z_2)$$

$$\int_{z_2}^{z_1} dw(w-z_2) \left(S_+^{(1\,\frac{3}{2}\,\frac{1}{2})}\right)^k \Psi_N^{(2)}(z_1, w, z_2) =$$

$$= \left(S_+^{10}\right)^k \int_{z_2}^{z_1} dw(w-z_2)\Psi_N^{(2)}(z_1, w, z_2) = r_N (S_+^{10})^k (z_1 - z_2)^{N+1}$$

$$r_N \sim ||\Psi_N||^2 \sim \gamma_N$$

$$\mathcal{R}(z_1,z_2) = z_{12}\int_{z_2}^{z_1}\frac{dw}{z_{12}}\int_{z_2}^{w}\frac{dw'}{z_{12}}\,\frac{w'-z_2}{z_1-w'}\left[\frac{1}{2}S_+\mathcal{O}_1(w,w') - (S_0-1)\mathcal{O}_2(w,w')\right],$$

$$\bar{\mathcal{R}}(z_1,z_2) = z_{12}\int_{z_2}^{z_1}\frac{dw}{z_{12}}\int_{z_2}^{w}\frac{dw'}{z_{12}}\,\frac{z_1-w}{w-z_2}\left[\frac{1}{2}S_+\mathcal{O}_1(w,w') - (S_0-1)\mathcal{O}_2(w,w')\right].$$

$$\mathcal{O}_1(w,w') = \left[i\mathbf{P}^\mu, \left[i\mathbf{P}_\mu, \mathcal{O}^{t=2}_{++}(w,w')\right]\right], \qquad \mathcal{O}_2(w,w') = \left[i\mathbf{P}^\mu, \frac{\partial}{\partial x^\mu}\mathcal{O}^{t=2}_{++}(w,w')\right].$$

$$\begin{aligned}\mathbb{A}(z_1,z_2) = &\frac{1}{4}\int_0^1 du\,\Big\{u^2\ln u\, z_1 z_2\,\mathcal{O}_1(z_1u,z_2u) + \Big(z_2\partial_{z_2} - \frac{z_1}{z_{12}} - \ln u\, z_2\partial^2_{z_2} z_{12}\Big)\mathcal{R}(uz_1,uz_2)\\ &- \Big(z_1\partial_{z_1} - \frac{z_2}{z_{21}} - \ln u\, z_1\partial^2_{z_1} z_{21}\Big)\bar{\mathcal{R}}(uz_1,uz_2)\Big\},\end{aligned}$$

$$\begin{aligned}\mathbb{B}(z_1,z_2) = &\frac{1}{8}\int_0^1\frac{du}{u^2}\Big\{u^2(1-u^2+u^2\ln u)z_1z_2\mathcal{O}_1(z_1u,z_2u)\\ &- \Big[(1-u^2)\Big(z_2\partial_{z_2} - \frac{z_1}{z_{12}}\Big) + (1-u^2+u^2\ln u)\, z_2\partial^2_{z_2} z_{12}\Big]\mathcal{R}(uz_1,uz_2)\\ &+ \Big[(1-u^2)\Big(z_1\partial_{z_1} - \frac{z_2}{z_{21}}\Big) + (1-u^2+u^2\ln u)\, z_1\partial^2_{z_1} z_{21}\Big]\bar{\mathcal{R}}(uz_1,uz_2)\Big\},\end{aligned}$$